Can Public Debt Crowd in Private Investment?*

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Abstract

Public debt can crowd in private, risky investment. While public debt leads to higher interest rates and thus requiring higher taxes, creating an excess burden, it improves households' ability to self-insure. It also enhances the safety of the average household's financial portfolio. In equilibrium, this encourages households to take on more risky, growth-promoting investments. We assess these channels using an incomplete markets endogenous growth model calibrated to U.S. data to revisit the question of what the optimal level of public debt is. Taking into account the growth effects of debt, our analysis suggests that a moderate increase in the debt-to-GDP ratio relative to its historical average is optimal, although only the richest households would benefit. The mechanism operating through economic growth is central to this result: absent growth effects, the welfare-maximizing level of public debt would be negative. A transition from the current debt-to-GDP ratio to its long-run optimal level is welfare-improving for all households. This is because the adjustment path induces a temporary increase of the growth rate that compensates poorer households for the utility losses they experience in the long run.

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1 Introduction

The multiple crises of recent decades have led to a sharp increase in public debt in most advanced economies. In some cases, this has sparked concerns about debt sustainability. More generally, concerns have been raised about the macroeconomic effects of the taxes needed to finance debt service, which can distort labor supply. Another significant argument against maintaining high levels of public debt is its potential to crowd out private investment.

However, at least since the seminal papers by Huggett (1993) and Aiyagari and McGrattan (1998), the economic literature has argued that positive or even elevated levels of government debt can still be desirable. They facilitate self-insurance by households and the economic value of this facilitated self-insurance can outweigh the negative effects of crowding out capital and distorting labor supply.[1] While the trade-offs involved are clear and important, the matter seems far from settled. What the literature has established is that it depends on details of the income process and the elasticity of labor supply, whether a high or a low debt is desirable.[2]

Importantly, almost all of the incomplete markets literature has used a framework of exogenous growth.[3] However, from an optimal policy point of view, this is an important restriction because the distortions and benefits at the level of activity can easily be dwarfed by the potential effects of government debt and taxation on endogenous growth. At the same time, the effects of public debt on growth are not clearly understood. On the one hand, the distortionary effect of taxation may discourage growth-enhancing investment. On the other hand, better-insured households may be more willing to take risks and invest in innovation (this line of argument dates back at least to Tobin, 1961, 1963), as more liquidity renders the downside of investment risk less painful and higher interest rates render the upside more enjoyable for longer.

To re-quantify these trade-offs between insurance, distortions and growth, we incorporate risky innovation and endogenous growth into an incomplete markets model of the type studied in Aiyagari and McGrattan (1998). We demonstrate that, for a wide range of debt levels, the growth promoting risk-tolerance effect dominates crowding out of capital investments and distortions in labor supply. Put differently, while pushing higher levels of public debt into the market still requires higher interest rates that are a fiscal burden and require higher distortionary taxes and crowd out capital, the higher interest rates, through the lens of the model, reflect better insurance and this

[1] Among others see Aiyagari and McGrattan (1998), Challe and Ragot (2010), Heathcote (2005), or Woodford (1990).
[2] See Krueger and Perri (2011), Röhrs and Winter (2015), Röhrs and Winter (2017), Bayer, Born and Luetticke (2023), and Dyrda and Pedroni (2023).
[3] An important exception is Krebs (2003) and Krebs, Kuhn and Wright (2015) who, however, do not focus on optimal government debt levels.

does enhance risk-taking via investment, promoting growth.

In our model, households face uninsurable income risks and self-insure against them. Unlike in the standard setting, households can invest not only in risk-free physical assets and/or government bonds, but also in risky, growth-enhancing projects. These projects are investments that, when successful, introduce new varieties to the economy. Each variety earns profits through monopolistic competition and expands the set of choices available to consumers, making the economy more efficient as a whole (see Melitz, 2003; Bilbiie, Ghironi and Melitz, 2012, the latter for a business cycle setting). Through an externality on the invention of new varieties (in the spirit of Romer, 1990), adding varieties promotes *growth* and does not only shift the *level* of productivity.

In detail, we assume that all households (that own varieties) operate a backyard enterprise which transforms raw goods into differentiated goods. This enterprise may fail periodically as its products become obsolete. In that case, the household forfeits all its investments in varieties and must start accumulating product varieties all over. Therefore, households must consider the trade-off between the returns on growth investments and the risks of these investments, as well as their need to self-insure through alternative assets. The government's supply of assets interacts with these trade-offs that households face. Higher levels of government debt facilitate self-insurance. Importantly, as we demonstrate, this enables households to take on more risk in their investments, thereby promoting growth.

To sharpen our understanding of the mechanisms, we first develop a stylized three-period version of our model before presenting the full quantitative framework. In the stylized model, ex ante identical households invest in productivity-enhancing risky assets or risk-free government bonds. Since the productivity-enhancing investments are developments of new varieties, households acquire only a part of the public return of the investment, and we have underinvestment in the *laissez-faire* equilibrium. This is exacerbated by the uninsurability of investment risk. After a successful initial investment, the investing household expects its return on the risky investment to decline over time. Therefore, it would like to save in a safe asset to smooth consumption. When liquidity is scarce, this is not possible and the effective return on the risky asset falls. As a result, when government debt is scarce, increasing the supply of government debt increases welfare because it increases the economic value of the return on the risky investment in good times.

Building on this intuition, we develop a quantitative heterogeneous agent model. In this framework, we find the well-known result of Aiyagari and McGrattan (1998) that higher levels of government debt crowd out capital and distort the supply of labor through the need for taxation. At the same time, we show that households, on average, are willing to invest more in risky projects. In a high-interest rate environment, they find it easier to build a buffer of wealth that helps them manage the risk of equity.

Calibrating the model to long-term U.S. averages for income risk, government debt, equity returns, and economic growth, we find that a 15 percent increase in government debt requires a 16-basis-point increase in the annual return on liquid assets. At this higher level of public debt and higher interest rates, households allocate more investments toward growth-enhancing projects boosting annual growth by 2.7 basis points. Our findings suggest that the optimal level of government debt is around 55% of annual GDP, which is within the range of optimal levels found by other authors (See for example Aiyagari and McGrattan, 1998; Flodén, 2001; Röhrs and Winter, 2017). At this level, the economy has a 2.8% smaller capital-to-output ratio, because of higher real interest rates and employs 1.6% less labor, due to higher labor income taxes. Yet, it achieves an annual growth rate that is 2.7 basis points higher compared to a 40% debt-to-GDP ratio.

In welfare terms, however, the crowding-in of innovation-driven growth outweighs the crowding-out effects on capital and labor. Measured by consumption equivalent variation (CEV), at the optimal debt-to-GDP ratio, households have, on average, a 0.17% higher welfare. However, this aggregate welfare gain is unequally distributed over wealth groups. While the bottom nine deciles along the wealth distribution lose from higher debt-to-GDP ratios, the top decile, and especially the top 1%, gains much and, on average, overcompensates the welfare losses of the poor. Since most households would suffer in utility terms, the long-run policy of increasing the debt-to-GDP ratio would be unpopular if households were to vote on it.

To better understand the individual channels that contribute to our result, we study a variant in which the government adjusts government expenditures instead of labor income taxation, and one variant with exogenous growth. We show that when abstracting from distortionary taxation and letting government expenditure adjust, the optimal debt level and the growth-enhancing effect of public debt would even be greater. Distortionary taxation is important for bounding our results. We also contrast our baseline result to a setting with exogenous growth, keeping all other model components identical. Without the endogenous growth channel, the optimal debt-to-GDP ratio would be negative, highlighting that the endogenous growth channel is the dominant driver of our welfare results.

Finally, we analyze the welfare implications of moving to the balanced growth path (BGP) with higher public debt and higher growth by studying a nonlinear perfect foresight transition. In contrast to the findings by other studies (Röhrs and Winter, 2017; Dyrda and Pedroni, 2023) we find that the transition to the higher debt-to-GDP ratio increases welfare for all households, such that welfare measured by the consumption equivalent variation increases by 0.6%. Concretely, the government reduces labor income taxes in the short-run, which redistributes wealth from wealthy to poor households. As poor households have a shorter planning horizon due to potentially being constrained, they value this upfront transfer and discount potential long-run welfare losses. Consequently, while in the long-run only the rich households benefit from the transition to the optimal balanced growth path with higher debt, in the short-run poor households are compensated. Hence, the policy of moving to this new balanced growth path would be favored by all households in the economy if households were to vote on it.

Related literature Our results contribute to three different strands of literature. First, they are related to the literature on heterogeneous agents in macroeconomic models that analyze fiscal policy outcomes (see, e.g., Woodford, 1990; Heathcote, 2005; Kitao, 2008; Challe and Ragot, 2010; Kaplan and Violante, 2014; McKay and Reis, 2016; Bayer, Born and Luetticke, 2023). To our knowledge, we are the first to integrate the analysis of fiscal policy within a heterogeneous agent model framework that includes portfolio choice and endogenous growth dynamics. Earlier contributions such as Krebs (2003) and Krebs, Kuhn and Wright (2015) also feature endogenous growth with heterogeneous agents through a portfolio choice to invest in human capital besides a risk-free asset. However, they do not focus on the role that the supply of the risk-free asset has on endogenous growth, but emphasize alternative factors.

We share similarities with the work of Gaillard and Wangner (2022). They use a model with heterogeneous agents and endogenous growth, but with the restriction that only an ex-ante fixed subset of households, "the innovators", contribute to aggregate growth, while all other households, "the workers", face idiosyncratic risk. Gaillard and Wangner (2022) emphasize the role of stabilization policy in reallocating resources between these different groups, thereby highlighting the trade-off between short-run demand stabilization and long-run growth stabilization. In contrast, our contribution diverges in its mechanism by emphasizing the critical interplay between idiosyncratic risk and growth-enhancing investment. Angeletos and Calvet (2006) and Modena and Regis (2024) analyze settings in which households have access to risky idiosyncratic background technology. Angeletos and Calvet (2006) studies the impact of interest rate changes on the economy. While they find that future higher interest rates depress investment today, we find the opposite. Modena and Regis (2024) illustrate how capital taxation financed by debt changes the wealth distribution, but do not consider optimal policy. Tamai (2026) studies the impact of public investment on growth and inequality, while we study the impact of public debt on these variables.

Second, we contribute to the literature examining the role of government debt in heterogeneous agent models. Following the seminal work of Aiyagari and McGrattan (1998), recent contributions have revisited the question of the optimal level of government debt and the optimal conduct of fiscal policy.[4] However, much of this literature

[4] For instance, see Flodén (2001), Krueger and Perri (2011), Gomes, Michaelides and Polkovnichenko

has primarily focused on determining the optimal level of public debt under the assumption of exogenous growth. Our analysis contributes to this literature by endogenizing the growth rate and quantifying the importance of liquidity provision and budget insurance in a framework that is consistent with both micro and macro data. While our focus on liquidity parallels that of Bayer, Born and Luetticke (2023), we enrich their model by incorporating an endogenous growth mechanism. This adds an important dimension to the endogenization of natural rates of interest in macroeconomic models (see, e.g., Bayer, Born and Luetticke, 2023, 2024; Bayer et al., 2019; Campos et al., 2024, for some recent contributions) and speaks also potentially to the effects of central banks changing the real rate on safe and liquid assets.

Finally, our work contributes to the endogenous growth literature (see, e.g., Romer, 1990; Aghion and Howitt, 1992; Kung and Schmid, 2015; Okada, 2022). The mechanism driving our main result is similar to the reduced-form equity financing shock used in Bianchi, Kung and Morales (2019) or the liquidity demand shock in Anzoategui et al. (2019). In both papers, these shocks lead to a reduction in the representative household's R&D investment. In our model, changes in households' insurance against idiosyncratic risk affect their demand for liquidity and risky equity. Thus, our model structure provides a microfoundation for the shocks used in the aforementioned papers. While papers within the entrepreneurial literature (see, e.g., Kitao, 2008; Buera and Shin, 2013; Midrigan and Xu, 2014; Buera, Kaboski and Shin, 2015) also incorporate heterogeneity, they primarily examine the impact of frictions on total factor productivity. While these models incorporate transitional dynamics, they do not feature endogenous long-run growth, which distinguishes our contribution.

The remainder of this paper is organized as follows. Section 2 develops a simple three period model to sharpen intuition. Section 3 develops the full infinite horizon model and describes the endogenous growth equilibrium. Section 5 illustrates the calibration of the model to US data, followed by the presentation of results from our policy experiments. Finally, Section 6 concludes. An Appendix follows.

2 A tractable analytical model

To form intuition, we first consider a three-period model of precautionary saving and portfolio choice.

(2012), Röhrs and Winter (2015), Bhandari et al. (2016), Röhrs and Winter (2017), Dyrda and Pedroni (2023), among others.

2.1 Model Description

In the first period, ex ante identical households decide to invest in a government bond or in the creation of new product varieties, a risky equity investment. In the second period, risk realizes and some households enjoy high returns on their equity investments, temporary monopoly rents on the varieties they created, while other households have to live exclusively on labor income. In the third period, the right to earn monopoly rents is extinguished and all households have only labor income. Since the right to monopoly rents vanished in the third period, the investing households do not earn the full economic rent from their investment, and thus there is a nonpecuniary externality. A government provides a risk-free asset and repays it by collecting lump-sum taxes.

Consumption, Investment and Portfolio Choices: Household utility depends only on consumption and we assume that the household's utility function is three times continuously differentiable and strictly concave. For simplicity, we abstract from discounting between periods one and two, but allow discounting of period three to capture the relative importance of the two future periods.

In period one, all households are endowed with ω goods. In addition, they receive government transfers (a negative lump-sum tax), $-\tau_1$. They decide to consume the endowment, c, to invest in government bonds, b_2, or to invest in the creation of new varieties, e.

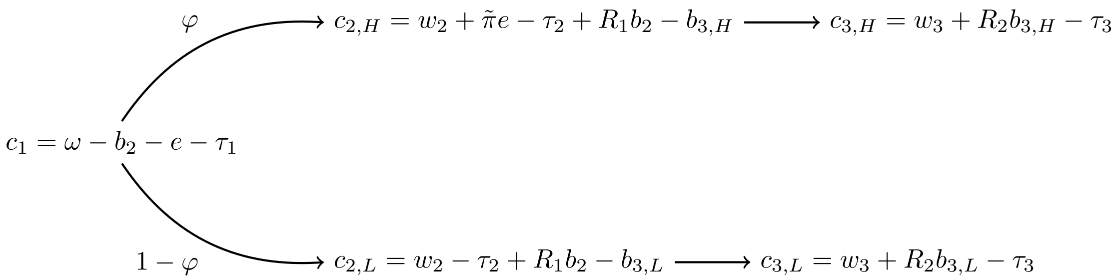
In period two, households are endowed with a unit of labor $N = 1$ from which they receive labor income w_2. In addition, government debt is repaid with gross interest R_1. For a fraction of φ households, the investment in the creation of varieties was successful ("H-state") and they receive monopoly rents $\tilde{\pi}$ for each unit they invested, in addition to their labor income. The other households ("L-state") receive only labor income. In period two, households divide their income between consumption c_2 and investments in government bonds b_2.

In period three, households are again endowed with a unit of labor $N = 1$ from which they receive labor income w_3. They no longer have any rights to monopoly rents, regardless of whether their investment in period two was successful or not. In neither period can households borrow in the risk-free asset (bonds) or equity. Figure 1 summarizes the timeline of the analytical model.

Production and factor incomes: A continuum of firms produces output using labor inputs with the production function $Y_t = \mathcal{E}N$. The efficiency of labor is proportional to the total risky investment that was successful, $\mathcal{E}$.

In period two, owners of varieties can claim a share $1 - \phi$ of the output, so that the per-unit profit is $\tilde{\pi} = (1 - \phi)Y_2/\mathcal{E}$, while the rest goes to labor income, implying a

Figure 1 Timeline of the analytical model



Note: Subscripts on consumption denote the respective period and household groups. b_2 denotes savings in the risk-free asset between periods one and two, while $b_{3,i}$ denotes savings between periods two and three of household group i. e denotes risky savings between periods one and two.

wage rate $w_2 = \phi Y_2$.[5] In the third period, all income is paid as wages to households, $w_3 = Y_3 = \mathcal{E}N$.

This implies that, in both periods, investment in risky assets increases factor incomes, but investors in varieties do not fully appropriate the economic rents they produce. In the full quantitative model, we micro-found this efficiency enhancing effect of investment in varieties through the preferences of variety-loving consumers.

Government: The government must balance its budget in each period. It issues $\mathcal{B}$ units of government bonds in period one, rolls these over in period two and pays them back in period three. In period one, the government pays out the proceeds of the government debt in the form of transfers. In periods two and three, it collects lump-sum taxes to finance the interest on the debt, i.e., the net interest rate in period two and the gross interest rate in period three, since all of the debt is then repaid.

$$\tau_1 = -\mathcal{B}, \quad \tau_2 = (R_1 - 1)\mathcal{B}, \quad \text{and } \tau_3 = R_2\mathcal{B}.$$

The government has no productive role except to provide liquidity to households.

Market Clearing: Market clearing requires that the asset, labor, and goods markets clear in all periods. The asset markets include the market for risk-free assets in periods one and two, and the market for risky assets in period one. The labor market exists in periods two and three, while the goods market exists in all three periods.

For the liquid asset market to clear, households must hold all government debt. Therefore, $\mathcal{B} = \int_0^1 b_{i,t} di$ must hold. In the first period this simplifies to $\mathcal{B} = b_2$ and $\mathcal{B} = \varphi b_{3,H} + (1 - \varphi) b_{3,L}$ in the second period. In period three, all debt is repaid.

Market clearing in the risky asset market requires $\mathcal{E} = \varphi \int_0^1 e_i di$, where e_i is the policy

[5] This payoff structure can be microfounded in a standard two-level production structure with symmetric intermediate goods producers enjoying monopolistic competition and a final goods bundler. The quantitative section provides a microfoundation for a production structure with labor and capital. The production structure obtained here is a special case without capital.

function in period one for the risky asset. Since all households are identical ex ante, this simplifies to $\mathcal{E} = \varphi e$.

Market clearing in the labor market requires that the labor demanded by the firm and used in production equals the labor supplied by households. Since households supply one unit of labor inelastically, labor market clearing requires $N_2 = N_3 = 1$.

Finally, goods market clearing requires that all endowments or produced goods are used by households. This yields the goods market clearing conditions, which we state in the order of the periods

$$\omega = c_1 + e, \text{ and } Y_t = \varphi c_{t,H} + (1-\varphi)c_{t,L} \text{ for } t = 2, 3.$$

Equilibrium: In equilibrium, all markets clear and households' consumption and investment decisions are optimal. This implies that government debt policy affects period one decisions only through its effect on future allocations and equilibrium prices, as $-\tau_1 = \mathcal{B}$.

2.2 Crowding in risky investment

In the following, we show that if government debt is small enough, an increase in government debt fosters investment in the risky asset. At zero debt, the H-type household expects income to fall between periods two and three, while the L-type household expects it to rise; the former wants to save, while the latter wants to borrow, but is strictly constrained to do so. By continuity, the same H-type-are-savers structure emerges even for small positive levels of government debt (see Appendix A for the formal argument).

Thus, for sufficiently small levels of debt, the H-type household buys up all government debt in period two, and we obtain as allocations for the H-type:

$$c_1 = \omega - e^* \tag{1}$$

$$c_{2,H} = w_2 + \tilde{\pi}e^* - \frac{1-\varphi}{\varphi}\mathcal{B} \tag{2}$$

$$c_{3,H} = w_3 + R_2\frac{1-\varphi}{\varphi}\mathcal{B}, \tag{3}$$

where e^* is the optimal risky investment. Since only in the H-state there is a positive payoff from investing in e, this investment is determined by the Euler equation

$$u'(c_1) = \varphi\tilde{\pi}u'(c_{2,H}) \quad \Leftrightarrow \quad u'(\omega - e^*) = \varphi\tilde{\pi}u'\left(w_2 + \tilde{\pi}e^* - \frac{1-\varphi}{\varphi}\mathcal{B}\right). \tag{4}$$

This immediately implies our

Proposition 1. *There exists $\mathcal{B}^*$ such that if $\mathcal{B} < \mathcal{B}^*$, the direct effect, i.e., keeping wages and profits fixed, of $\mathcal{B}$ on optimal risky investment e^* is positive.*

Proof. Taking the total differential of (4), but keeping $\mathcal{E}$ fixed (partial equilibrium), we obtain

$$-u''(\omega - e^*)de^* = \varphi\tilde{\pi}u''\left(w_2 + \tilde{\pi}e^* - \frac{1-\varphi}{\varphi}\mathcal{B}\right)(\tilde{\pi}de^* - \frac{1-\varphi}{\varphi}d\mathcal{B})$$

which implies for the partial equilibrium effect of a change in bonds on risky investment, assuming a fixed number of varieties:

$$\left.\frac{de^*}{d\mathcal{B}}\right|_{\mathcal{E}} = \frac{1-\varphi}{\varphi}\frac{\varphi\tilde{\pi}u''\left(w_2 + \tilde{\pi}e^* - \frac{1-\varphi}{\varphi}\mathcal{B}\right)}{u''(\omega - e^*) + \varphi\tilde{\pi}^2u''\left(w_2 + \tilde{\pi}e^* - \frac{1-\varphi}{\varphi}\mathcal{B}\right)} > 0 \tag{5}$$

where the last inequality follows from $u''(\cdot) < 0$. □

The intuition for this result is simple: with more government debt, the investor expects to be able to smooth the returns on her investment better over periods two and three (from the point of view of an individual investor that looks at prices instead of market clearing quantities: the interest rate between periods two and three is higher). This increases the marginal value of the returns to the investment in the case of success. In this sense, our result shows some structural similarity to the point made by Woodford (1990) that government debt can crowd in investment when liquidity is scarce.

However, if all households invest more in the risky asset, this changes output in period two as $\mathcal{E} = \varphi e^*$ and $Y_2 = \mathcal{E}$. Households that are richer in period two will save and invest less in period one. Yet, the direct effect dominates this indirect one as we show in

Proposition 2. *For $\mathcal{B} < \mathcal{B}^*$ (from Proposition 1), the indirect effect of a change in government debt through output (wages and profits) dampens but does not overturn the direct effect, so the total effect on risky investment remains positive.*

Proof. First, we observe that the per-variety profit is $\tilde{\pi} = (1-\phi)Y_2/\mathcal{E} = (1-\phi)$ and thus independent of the total number of varieties, $\mathcal{E}$. The overall effect of a change in government bonds on the investment into the risky asset is therefore the sum of the direct effect and the indirect effect through a wage change in period two:

$$de^* = \left.\frac{de^*}{d\mathcal{B}}\right|_{\mathcal{E}} d\mathcal{B} + \left.\frac{de^*}{dw_2}\right|_{\mathcal{E}} dw_2 = \left.\frac{de^*}{d\mathcal{B}}\right|_{\mathcal{E}} d\mathcal{B} + \left.\frac{de^*}{dw_2}\right|_{\mathcal{E}} \phi\varphi de^*$$

$$\implies \frac{de^*}{d\mathcal{B}} = \left.\frac{de^*}{d\mathcal{B}}\right|_{\mathcal{E}}\left[1 - \phi\varphi\left.\frac{de^*}{dw_2}\right|_{\mathcal{E}}\right]^{-1}$$

Here $\phi\varphi = \frac{dw}{d\mathcal{E}}\frac{d\mathcal{E}}{de^*}$. The total effect $\frac{de^*}{d\mathcal{B}}$ is positive, if $\left.\frac{de^*}{dw_2}\right|_{\mathcal{E}}\phi\varphi < 1$.

Next, we calculate the effect of a wage change in period two on investment in period one, for otherwise fixed prices, as we did for the effect of a change in government debt and obtain

$$\left.\frac{de^*}{dw_2}\right|_{\mathcal{E}} = -\frac{\varphi\tilde{\pi}u''\left(w_2 + \tilde{\pi}e^* - \frac{1-\varphi}{\varphi}\mathcal{B}\right)}{u''(\omega - e^*) + \varphi\tilde{\pi}^2 u''\left(w_2 + \tilde{\pi}e^* - \frac{1-\varphi}{\varphi}\mathcal{B}\right)} < 0, \tag{6}$$

where the last inequality follows from $u''(\cdot) < 0$. □

In Appendix A, we derive the allocations of households for arbitrary debt levels and derive the threshold debt level $\mathcal{B}^*$ as function of the parameter values. Finally, for utility being logarithmic $u(c) = \log c$, we derive stronger statements about crowding-in and welfare. We show that an increase of government debt that crowds in risky investment is also welfare increasing. The intuition for this positive welfare effect is that government debt is neutral in period 1, and the tax to repay period 1 debt (cum interest) does not create any redistribution. The reissuance of government debt in period 2 allows the L-type to effectively borrow against its future income and at an interest rate lower than the type's discount rate. Similarly, the H-type can smooth its consumption more because it is less constrained to save. The additional investment in the varieties adds to the welfare outcome because it generates an externality on income in periods two and three. This argument holds as long as the economy is not Ricardian, that is government debt alters the interest rate and/or consumption allocations.

The model in this section is kept simplistic on purpose. To illustrate the key mechanism we kept the environment Ricardian while abstracting from income and ex-ante wealth heterogeneity. We address these important aspects of reality in the next section.

3 Quantitative model

Next, we extend the model to a fully dynamic setting, an economy with incomplete markets and endogenous growth with infinitely lived agents. In this economy, we allow taxes to distort factor incomes and in particular labor supply. We focus on its stationary equilibrium around the endogenous growth path before then studying the transition between balanced growth paths.

The production side embeds a model of expanding-variety innovation into an otherwise standard incomplete-markets model. Instead of assuming an exogenous growth process, we assume that new varieties can be added to the economy in the tradition of Romer (1990). Each variety is sold under monopolistic competition and earns rents that accrue to the entrepreneurial households operating it. Beyond these intermediate inputs, capital and labor also enter production.

Households face idiosyncratic labor-productivity risk. Markets are incomplete, but households self-insure. They can invest in physical capital, government bonds or acquire varieties to start or expand a business (equity). In doing so, they solve a portfolio choice problem, because equity investment carries non-diversifiable idiosyncratic risk, while capital and government debt are perfectly diversified and riskless.

In addition to the firm and household sectors, we model a fiscal authority that levies taxes on labor income, asset payoffs, and profits, provides lump-sum transfers, issues government bonds, and uses tax revenue for (wasteful) government consumption.

We begin by describing the production sector of the model, before turning to a detailed description of the household side of the modified growth economy and the government.

3.1 Firms

There are four production sectors. Final goods are produced under perfect competition using physical capital, K_t, labor L_t, and a composite of intermediate inputs Q_t. This composite is produced by bundling of differentiated varieties Q_{jt} among which there is monopolistic competition.

These varieties are produced by entrepreneur households in some form of backyard enterprise, differentiating final goods. Not all households can offer all diversified goods. Each household i has access to its set e_{it} of varieties that it can produce. This renders the set of all varieties offered, $\mathcal{E}_t$, endogenous and partitioned by the varieties each household i offers e_{it}.

An innovation sector produces ideas for new varieties that households can buy and irreversibly add to their backyard production technology. Next, we discuss the production structure in more detail.

3.1.1 Intermediate goods

A competitive bundler buys differentiated goods that each household produces with its backyard technology and bundles them using the production function

$$Q_t \equiv \left[\int Q_{it}^{\frac{\eta-1}{\eta}} di \right]^{\frac{\eta}{\eta-1}}. \tag{7}$$

where i indexes the unit mass of households. In producing the goods sold to the bundler, household i has access to the production of e_{it} varieties such that Q_{it} is itself a bundle:

$$Q_{it} \equiv \left[\int_0^{e_{it}} Q_{ijt}^{\frac{\epsilon-1}{\epsilon}} dj \right]^{\frac{\epsilon}{\epsilon-1}}, \tag{8}$$

where Q_{ijt} is the quantity of good j produced by household i at time t.

For simplicity, we assume that both substitution elasticities are equal, $\eta = \epsilon > 1$, so that we can write the aggregate bundling technology more compactly in terms of individual goods that households produce

$$Q_t = \left[\int_0^{\mathcal{E}_t} Q_{jt}^{\frac{\epsilon-1}{\epsilon}} dj \right]^{\frac{\epsilon}{\epsilon-1}} . \tag{9}$$

Here Q_{jt} is now the quantity offered in some variety j without specifying the household offering it, and $\mathcal{E}_t = \int e_{it} di$ is the measure of varieties available in the economy.

The bundler buys each individual variety at the price P_{jt}. Minimizing the cost $\int_0^{\mathcal{E}_t} P_{jt} Q_{jt} dj$ to produce Q_t, she will therefore demand of each individual variety

$$Q_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\epsilon} Q_t, \tag{10}$$

with $P_t = \left(\int_0^{\mathcal{E}_t} P_{jt}^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}$ as the price of the composite good. Equation (10) denotes the demand for each individual variety Q_{jt}.

Conversely, this implies that a constant mark-up is charged on each variety. Using the final good as the numeraire, we get

$$P_{jt} = \frac{\epsilon}{\epsilon - 1} \quad \forall j, t. \tag{11}$$

Since all households charge the same price, all quantities are identical and households make the same profit on each variety:

$$\pi_{jt} = \frac{1}{\epsilon - 1} Q_{jt}. \tag{12}$$

Using the fact that one unit of the final good is used to produce one unit of the intermediate good Q_{jt}, we can also express the production of Q_t in terms of the final good $X_t = \int Q_{jt} dj$ used in its production as

$$Q_t = \mathcal{E}_t^{\frac{1}{\epsilon-1}} X_t, \tag{13}$$

where $\mathcal{E}_t^{\frac{1}{\epsilon-1}}$ reflects the productivity-enhancing aspect of adding varieties.

3.1.2 Final Goods Producer

Final goods are produced by a representative firm using capital K_t, labor N_t, and intermediate goods Q_t according to the gross output production technology.

$$Z_t = \bar{A} \left(K_t^{\alpha} N_t^{1-\alpha} \right)^{1-\nu} Q_t^{\nu}. \tag{14}$$

α and ν are the share of capital and the share of intermediate goods in production, $\bar{A}$ is a scaling for productivity. We can use equation (13) to translate this output production function into value added in final goods production $Y_t = Z_t - X_t$. From the optimality condition for intermediate inputs:

$$P_t Q_t = \frac{\epsilon}{\epsilon - 1} X_t = \nu Z_t \tag{15}$$

follows $Y_t = \left(1 - \nu \frac{(\epsilon-1)}{\epsilon}\right) Z_t$ and after some algebra as well as normalizing the productivity scale $\bar{A}$ in the right way, we get

$$Y_t = K_t^{\alpha} N_t^{1-\alpha} \mathcal{E}_t^{\overline{\frac{\nu}{(\epsilon-1)(1-\nu)}}}, \tag{16}$$

which, under the further assumption $1 - \alpha = \frac{\nu}{(\epsilon-1)(1-\nu)}$, we can write as

$$Y_t = K_t^{\alpha} (\mathcal{E}_t N_t)^{1-\alpha}. \tag{17}$$

The latter assumption is necessary to obtain a balanced growth path.

Because of monopolistic competition between intermediate goods, only a fraction of $\phi = \frac{\epsilon(1-\nu)}{\epsilon(1-\nu)+\nu}$ of this value added compensates capital and labor as factors of production, and the remainder, $1 - \phi$, becomes entrepreneurial income.

Let w_t, r_t and δ refer to the wage rate, the interest rate and depreciation. With these definitions, the usual first-order conditions determine factor demands/factor prices

$$r_t + \delta = \phi\alpha \frac{Y_t}{K_t} = \phi\alpha \left(\frac{\mathcal{E}_t N_t}{K_t}\right)^{1-\alpha}, \tag{18}$$

$$w_t/\mathcal{E}_t = \phi(1-\alpha)\frac{Y_t/\mathcal{E}_t}{N_t} = (1-\alpha)\left(\frac{\mathcal{E}_t N_t}{K_t}\right)^{-\alpha}, \tag{19}$$

$$\text{and } \pi_t = (1-\phi) Y_t. \tag{20}$$

The last equation determines the total profit from the production of goods, which is a fixed proportion of output.

3.1.3 Innovation Sector

Varieties are invented by a continuum of perfectly competitive innovators. The representative innovator does so by conducting research, using R&D expenditures S_t (in terms of final goods). The representative innovator produces new varieties Δ_t according to the linear production function

$$\Delta_t = \chi_t S_t, \tag{21}$$

where χ_t is the productivity of the innovation sector. Given that there is a perfectly competitive continuum of identical innovators, the representative innovator takes χ_t as given even if there is an externality as in Comin and Gertler (2006)

$$\chi_t = \chi \frac{\mathcal{E}_t}{\mathcal{E}_t^{\rho} S_t^{1-\rho}}. \tag{22}$$

Here, χ and $0 \leq \rho \leq 1$ are scalars that control the importance of investment in ideas for growth. As in Romer (1990), there is a positive spillover of the aggregate stock of varieties $\mathcal{E}_t$ on individual productivity. However, we also model congestion externalities via the factor $\mathcal{E}_t^{\rho} S_t^{1-\rho}$. This externality increases the cost of developing new varieties as the aggregate R&D intensity, S_t, increases. The smaller ρ, the less important is the endogenous growth mechanism, the higher χ, the faster the economy grows.

Under this functional assumption, it is straightforward to show that, in equilibrium with symmetric innovators, the variety elasticity of R&D expenditure becomes ρ. To ensure that the growth rate of new intermediate products is stationary, we assume that the congestion effect is positively correlated with the existing number of varieties, denoted by $\mathcal{E}_t$. This is tantamount to assuming that, all else equal, the marginal return to R&D investment declines as the economy becomes more sophisticated, as measured by the number of varieties.[6]

Households acquire varieties in an innovation market. We will discuss the details when describing the household's consumption-savings problem. Importantly, in the market for product ideas, innovators can sell new varieties at a price of q_t. Innovators will continue to add varieties until the marginal cost of a new variety equals the price, q_t. Assuming free entry and perfect competition, the price of new varieties is fixed at the marginal cost of production. Expressed in terms of new and existing varieties and output, this gives the price per variety

$$q_t = \chi^{-\frac{1}{\rho}} \left(\frac{\Delta_t}{\mathcal{E}_t} \right)^{\frac{1-\rho}{\rho}}. \tag{23}$$

3.1.4 Per-Variety Notation

Later we want to solve the model around a balanced growth path. To do this, we express the capital stock and output relative to the number of varieties, denoting the

[6] Endogenous growth models such as Romer (1990), which use labor as the only input factor in R&D production, also feature procyclical R&D costs. The relevant cost of producing a unit of new varieties is the real wage rate, and the wage rate is procyclical in these models.

per-variety variables by tilda, i.e. $\tilde{K}_t = K_t/\mathcal{E}_t$ and $\tilde{Y}_t = Y_t/\mathcal{E}_t$. This gives us factor prices:

$$r_t + \delta = \phi\alpha \frac{\tilde{Y}_t}{\tilde{K}_t} = \phi\alpha \left(\frac{\tilde{K}_t}{N_t}\right)^{\alpha-1}, \tag{24}$$

$$\tilde{w}_t = \phi(1-\alpha)\frac{\tilde{Y}_t}{N_t} = \phi(1-\alpha)\left(\frac{\tilde{K}_t}{N_t}\right)^{\alpha}, \tag{25}$$

$$\text{and } \tilde{\pi}_t = (1-\phi)\tilde{Y}_t. \tag{26}$$

The last equation is the profit per variety.

3.2 Households

The household side is similar to the setup in Aiyagari and McGrattan (1998): households face idiosyncratic income risk and self-insure against it. We extend the asset market by an illiquid investment option into a risky asset. We model the illiquidity as in Bayer, Born and Luetticke (2024) as a random market participation. The illiquid asset is risky because the entire amount invested can be lost in any period. As before, all variables with a tilde are expressed relative to the total number of varieties in the economy $\mathcal{E}_t$.

3.2.1 Productivity, Preferences, and Income

There is a continuum of households $i \in [0, 1]$. Households earn income from work, they earn interest income on their financial assets (consisting of claims on physical capital and bonds), and they earn profit income from their entrepreneurial activities.

They face risks in these activities as well as in the labor market. We model the latter as persistent fluctuations in a household's human capital h_{it}. This means, we focus on long-term labor market risks rather than, say, the risk of unemployment and thus assume human capital evolves according to

$$\log h_{it} = \rho_h \log h_{it-1} + \epsilon_{it}. \tag{27}$$

Here, ϵ_{it} are normally distributed shocks with variance σ_ϵ^2 and mean $\mu_\epsilon = -\frac{\sigma_\epsilon^2}{2(1+\rho_h)}$. This implies a (normalized) average productivity of unity: $\mathbb{E}(h_{it}) = 1$.

As described in the last subsection, we assume that in addition to offering labor, each household also engages in some entrepreneurial activity, offering a range of intermediate inputs. The number of varieties offered by household i is denoted by e_{it}. We think of these varieties as distinct but related products. Occasionally, the range of products offered by household i becomes obsolete. The household then loses all of its varieties, and the household must start over by accumulating new ideas of varieties. This obsolescence of the varieties offered by the household renders ideas investments risky. Moreover, we assume that investing in ideas is only possible from time to time. This makes them illiquid and implies that some households do not produce any varieties at all if they never had the opportunity and resources to invest in them.

Households have time-separable King, Plosser and Rebelo (1988) (KPR) type preferences and derive felicity from consuming the final good c_{it} and disutility from supplying labor n_{it}. Households discount felicity with a time discount factor β and maximize the discounted sum

$$V = \mathbb{E}_0 \max_{\{c_{it}, n_{it}, a_{it}, e_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\ln(c_{it}) - \omega \frac{n_{it}^{1+\gamma}}{1+\gamma} \right]$$

where ω is a scaling parameter that determines average labor supply and γ is the inverse of the Frisch elasticity. The preference specification allows us to recast the household planning problem as a choice over labor supply and per-variety consumption $\tilde{c}_{it}$.

$$\mathbb{E}_0 \max_{\{\tilde{c}_{it}, n_{it}, \tilde{a}_{it}, \tilde{e}_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\ln(\tilde{c}_{it}) + \ln(\mathcal{E}_t) - \omega \frac{n_{it}^{1+\gamma}}{1+\gamma} \right], \tag{28}$$

which allows us later to solve the planning problem in stationary recursive form.

We assume a linear tax schedule for all income, so that a household's net labor income is given by

$$y_{it} = (1 - \tau_t^L) \tilde{w}_t n_{it} h_{it} \mathcal{E}_t, \tag{29}$$

where $\tilde{w}_t$ is the wage rate per variety, as derived in the last section, and τ_t^L is the linear labor tax. Given net labor income, the first-order condition on labor supply implies the optimal ratio of

$$n_{it}^{\gamma} = \frac{(1 - \tau_t^L) \tilde{w}_t h_{it}}{\tilde{c}_{it} \omega}, \tag{30}$$

which defines the optimal supply of labor. In addition to receiving income from labor, households receive after-tax profit income from their entrepreneurial activities, after-tax asset income from capital and bonds, and potentially (non-distortionary) transfers from the government $\mathcal{T}_{it} = \mathcal{T}_t(h_{it})$. Since each variety e_{it} earns the same gross profit $\tilde{\pi}$ and each unit of wealth a_{it} earns the gross return r_t, a household's total after-tax income is given by

$$\left((1 - \tau_t^a) r_t \tilde{a}_t + (1 - \tau_t^{\pi}) \tilde{\pi}_t \tilde{e}_{it} + \tilde{y}_{it} + \tilde{\mathcal{T}}_{it} \right) \mathcal{E}_t,$$

where $\tilde{e}_{it}$ is the number of varieties of household i relative to the average number in the economy.

3.2.2 Household Maximization Problem

Households maximize utility (28) choosing consumption ($\tilde{c}_{it}$) and their portfolio choice in terms of liquid asset holdings $\tilde{a}_{it+1}$ and ideas/varieties, equities in the following, $\tilde{e}_{it+1}$

given their after tax incomes (29) and the budget and borrowing constraint

$$\tilde{c}_{it} + \tilde{a}_{it+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} + q_t\tilde{e}_{it+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} = R_t\tilde{a}_{it} + (q_t + (1-\tau_t^\pi)\tilde{\pi}_{it})\tilde{e}_{it} + \tilde{y}_{it} + \tilde{\mathcal{T}}_{it}, , \tag{31}$$

$$\tilde{a}_{it+1} \geq 0, \tilde{e}_{it+1} \geq 0.$$

where $\tilde{a}_{it}$ is real wealth and $R_t = 1 + (1-\tau_t^a)r_t$ is the after-tax return on real wealth.

To account for the fact that most households are not entrepreneurs, we assume that only a random fraction of households, denoted by the parameter λ, can trade equities in a given period. The remaining $(1-\lambda)$ households will only be able to actively adjust their portfolio of liquid assets. However, we assume that a portion of profits is reinvested automatically to maintain the relative share of equities, $\tilde{e}_{it}$, despite trend growth in the absence of active portfolio adjustment.[7]

Thus, households face a trade-off: they can save in the liquid asset, $\tilde{a}_{it+1}$, to insure against idiosyncratic income risk or invest in high-return, high-risk equity $\tilde{e}_{it+1}$. We model the risk of investing in equity by assuming that, with probability $1-\varphi$, all of the equity held by the household is lost each period because the household's portfolio of assets becomes obsolete. Therefore, equity investment carries idiosyncratic risk that cannot be diversified due to the existing market structure. These two assumptions together make equity a profitable but illiquid and risky investment.

Since a household's decisions will be nonlinear functions of its wealth a_{it}, its equities e_{it}, and its productivity h_{it}, all prices will be functions of the joint distribution Θ_t of (a_{it}, e_{it}, h_{it}) in t. This makes Θ_t a state variable of the household's planning problem, leaving us with three functions that characterize the household's problem: the value functions V_t^a and V_t^n for the two cases illustrated above, and the continuation value $\mathbb{W}_{t+1}$. Omitting individual indices and letting variables with prime denote the next period value, we can characterize these value functions in the following equations:

$$V_t^a(\tilde{a},\tilde{e},h) = \max_{\tilde{a}',\tilde{e}'} u[\tilde{c}(\tilde{a},\tilde{a}',\tilde{e},\tilde{e}',h), n(\tilde{a},\tilde{a}',\tilde{e},\tilde{e}',h)] + \beta\mathbb{W}_{t+1}(\tilde{a}',\tilde{e}',h) \tag{32}$$

$$V_t^n(\tilde{a},\tilde{e},h) = \max_{\tilde{a}'} u[\tilde{c}(\tilde{a},\tilde{a}',\tilde{e},\tilde{e},h), n(\tilde{a},\tilde{a}',\tilde{e},\tilde{e},h)] + \beta\mathbb{W}_{t+1}(\tilde{a}',\tilde{e},h) \tag{33}$$

$$\begin{aligned}\mathbb{W}_{t+1}(\tilde{a},\tilde{e},h) =& \varphi\Big(\lambda\mathbb{E}_t[V_{t+1}^a(\tilde{a},\tilde{e},h')] + (1-\lambda)\mathbb{E}_t[V_{t+1}^n(\tilde{a},\tilde{e},h')]\Big)\\ &+ (1-\varphi)\Big(\lambda\mathbb{E}_t[V_{t+1}^a(\tilde{a},0,h')] + (1-\lambda)\mathbb{E}_t[V_{t+1}^n(\tilde{a},0,h')]\Big)\end{aligned} \tag{34}$$

Expectations about the continuation value are made with respect to the productivity state h_{it} conditional on the current states. Maximization is subject to the appropriate budget and borrowing constraints (31).

[7] This assumption simplifies the numerics significantly and means that in choosing only the liquid asset, we do not need to interpolate off grid values for equity.

3.3 Government

The government operates a fiscal authority that issues government bonds B_t to finance deficits, chooses tax rates in the economy, provides transfers to households, and has (wasteful) government consumption. It is thus summarized by the budget equation (in per-variety notation)

$$\tilde{G}_t + \tilde{\mathcal{T}}_t + R_t\tilde{B}_t = \tilde{B}_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} + \tilde{T}_t. \tag{35}$$

The government is assumed to run a budget deficit and chooses debt $\tilde{B}_{t+1}$. Besides issuing bonds, the government uses tax revenues $\tilde{T}_t$, defined below, to finance government expenditures $\tilde{G}_t$ and interest on debt $(R_t^b - 1)\tilde{B}_t$. The government taxes all incomes such that total taxes are $\tilde{T}_t = \mathbb{E}_{it}(\tau_t^L \tilde{w}_t h_{it} n_{it} + \tau_t^a r_t \tilde{A}_t + \tau_t^\pi \tilde{\pi}_t)$, where $\mathbb{E}_{it}$ is the cross-sectional average. In the baseline, the government adjusts the tax rate τ_t^L to satisfy the budget constraint.

3.4 Aggregates, Growth, Market Clearing, and Equilibrium

This section deals with the aggregates in this economy and their behavior. We begin by describing all the aggregates, illustrate the determinants of the growth rate in the economy, continue by specifying the market clearing conditions, and finally define the dynamic equilibrium in the economy.

3.4.1 Aggregation

Each household holds assets $\tilde{a}_{it}$ and shares $\tilde{e}_{it}$. We define the corresponding aggregates:

$$\tilde{A}_t = \int_0^1 \tilde{a}_{it} di \quad \text{and} \quad \mathcal{E}_t = \int_0^1 \tilde{e}_{it} di \tag{36}$$

Similarly, the aggregate effective labor supply is

$$N_t = \int_0^1 h_{it} n_{it} di. \tag{37}$$

Having defined the aggregates, we can define the law of motion for capital

$$\tilde{K}_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} = (1-\delta)\tilde{K}_t + \tilde{I}_t, \tag{38}$$

where $(1 - \delta)$ is the fraction of the capital stock that has not been depreciated and I_t refers to the investment in total physical capital.

3.4.2 Aggregate Growth

The economy features endogenous growth through the endogenous accumulation of equity $\tilde{e}_{it}$ by households. In each period, the varieties of $1-\varphi$ households become obsolete and these households lose all their equity. Thus, only φ households remain with their stock of $\tilde{e}_{it}$ in the next period. Thus, we can write the law of motion of the total number of varieties $\mathcal{E}_t$ as

$$\mathcal{E}_{t+1} = \varphi \mathcal{E}_t + \Delta_t,$$

where Δ_t is the new number of varieties as in equation (21). φ is the fraction of varieties that have not become obsolete. This implies the following expression for the growth rate of new technologies:

$$g_{t+1} := \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} - 1 = \frac{\Delta_t}{\mathcal{E}_t} - (1-\varphi). \tag{39}$$

Having established the aggregates and the growth rate, we can define market clearing in all markets.

3.4.3 Market clearing

The labor market clears at the competitive wage given in (25) with aggregate labor as in (37). The asset market clears whenever

$$\tilde{B}_{t+1} + \tilde{K}_{t+1} = \tilde{A}^d(R_t, \tilde{\pi}_t, \tilde{w}_t, q_t, \tau_t^L, \Theta_t) := \mathbb{E}_{it}[\lambda \tilde{a}^*_{a,t} + (1-\lambda)\tilde{a}^*_{n,t}]$$

holds, where $\tilde{a}^*_{a,t}$, and $\tilde{a}^*_{n,t}$ are policy functions of the states $(\tilde{a}_{it}, \tilde{e}_{it}, h_{it})$, and depend on the current set of prices and tax rates $(R_t, \tilde{\pi}_t, \tilde{w}_t, q_t, \tau_t^L)$. $\tilde{K}_{t+1}$ denotes the supply of assets from the firm side such that the left-hand side above represents the total supply of liquid assets in which households can save. Expectations on the right-hand-side expression are taken over the distribution Θ_t. An equilibrium requires the total net amount of assets households demand $\tilde{A}^d$ to equal the supply of government bonds $\tilde{B}_{t+1}$ and the supply of capital $\tilde{K}_{t+1}$. To ensure the market clearing, the interest rate on liquid assets R_t adjusts.

The market for equities/varieties clears when

$$\tilde{\Delta}_t = \frac{\mathcal{E}_{t+1} - (1-\varphi)\mathcal{E}_t}{\mathcal{E}_t} = \mathbb{E}_t[\lambda \tilde{e}^*_{a,t} + (1-\lambda)\tilde{e}_{it}] - (1-\varphi).$$

The first expression on the left side of the equation (3.4.3) is the supply of new varieties, whereas the expression on the right determines households demand for new varieties. For the market of new varieties to clear, the price of buying a new variety q_t adjusts.

Finally, the goods market clears when

$$Y_t = C_t + I_t + G_t + S_t. \tag{40}$$

The goods market clears due to Walras' law, whenever the labor, capital, equity, and bond markets clear.

3.4.4 Dynamic equilibrium

At the beginning of each period, agents are heterogeneous in three dimensions summarized by the state vector $\tilde{s}_{it} = (\tilde{a}_{it}, \tilde{e}_{it}, h_{it})$, i.e. asset holdings $\tilde{a}_{it}$, equity holding $\tilde{e}_{it}$, and labor productivity h_{it}. An equilibrium in this model features sequences of prices $\mathcal{P} = \{g_t, R_t, w_t, q_t, \pi_t, \tau_t^L, \tau_t^K, \tau_t^\pi\}_{t=0}^\infty$, sequences of capital, and labor $\{K_t, N_t\}_{t=0}^\infty$, sequences of policy functions $\{\tilde{c}^*_{a,t}, \tilde{c}^*_{n,t}, \tilde{a}^*_{a,t}, \tilde{a}^*_{n,t}, \tilde{e}^*_{a,t}, n^*_{a,t}, n^*_{n,t}\}_{t=0}^\infty$, value functions $\{\tilde{V}_t, \tilde{V}_t^n, \tilde{\mathbb{W}}_{t+1}\}_{t=0}^\infty$, a law of motion $\Gamma_{\mathcal{P}_t}$, and a sequence of distributions $\{\Theta_{\mathcal{P}_t}\}_{t=0}^\infty$ over individual asset holdings, equity, and productivity, such that

1. The policy functions $\{\tilde{c}^*_a, \tilde{c}^*_n, \tilde{a}^*_a, \tilde{a}^*_n, \tilde{e}^*_a, n^*_a, n^*_n\}$ solve the households' planning problem given prices and the continuation values $\tilde{\mathbb{W}}$.
2. Together with the transition matrices of the exogenous states s, the policies induce a law of motion $\Gamma_{\mathcal{P}}$.
3. The distribution solves the forward equation $\Theta_{\mathcal{P}_{t+1}} = \Gamma_{\mathcal{P}_t}(\Theta_{\mathcal{P}_t})$.
4. The value functions $\{\tilde{V}_t, \tilde{V}_t^n, \tilde{\mathbb{W}}_{t+1}\}$ solve the equations (32), (33), and (34) in every period.
5. The labor, the final goods, the market for equities, and the asset market clear in every period.
6. The interest rate clears the asset market, returns on capital are determined by the marginal product of capital, the wage rate is determined as the marginal product of labor, profits are determined by the optimal behavior of the intermediate producer, the price of new varieties is determined by the optimality condition of the innovator, and the labor tax rate adjusts to clear the government budget constraint.
7. Capital $\tilde{K}_t$ accumulates according to eq. (38), and the growth rate is determined as in eq. (39)

First, we solve the economy around a balanced growth path, where all detrended aggregate variables are constant over time. Hence, we solve for the steady state of the detrended economy. Thereafter, we solve for the nonlinear, perfect-foresight transition of the economy between different balanced growth paths.

4 Calibration of the Quantitative Model

We calibrate the model to the U.S. economy with one period representing a year. The model's parameters are either drawn from standard values commonly used in the literature or calibrated to match key targets in the baseline balanced growth path. Table 1 presents the parameters for households, firms, and the government.

On the household side, we follow Bayer, Born and Luetticke (2023) and target a capital-to-output ratio of $K_t/Y_t = 2.875$ and a debt-to-output ratio of $B_t/Y_t = 0.4$. We calibrate the discount rate β such that households supply the sum of these components, that is the asset-to-output ratio is $(K_t + B_t)/Y_t = 3.275$. We use a standard value of $1/2$ for the Frisch elasticity based on Chetty et al. (2011), such that $\gamma = 2$. We calibrate the adjustment probability for varieties to $\lambda = 5.86\%$ to match the average U.S. wealth Gini of 0.82 from 1954 until 2019. Moreover, we calibrate the scalar on labor disutility ω to normalize labor supply to unity $N_t = 1$. The income process parameter $\rho_h = 0.95$ is taken from Storesletten, Telmer and Yaron (2004), while we calibrate $\sigma_h = 0.23$ to match the cross-sectional variance of log labor earnings of 0.84 as in Song et al. (2019).

On the firm side, we set the elasticity of substitution between different varieties $\epsilon = 3.8$, as in Bilbiie, Ghironi and Melitz (2012) and in the ballpark of Bachmann et al. (2023). We then set the share of intermediate inputs $\nu = 0.6$, such that the ratio of intermediate inputs to gross output X_t/Z_t equals 0.45, consistent with historical U.S. data.[8] These values for the substitution elasticity and the share of intermediate inputs give us a share of entrepreneurial and labor incomes of 66%.[9] We follow Bayer, Born and Luetticke (2023) and assume a yearly depreciation rate of $\delta = 7\%$. Finally, we set all tax rates to $\tau^L = \tau^K = \tau^\pi = 25.7\%$ to satisfy the government budget constraint with an average government expenditure to GDP ratio of $G/Y = 0.2$ as in Bayer, Born and Luetticke (2023).

The most non-standard part of our calibration for an incomplete markets model regards the production of varieties/ideas. We set the probability of a variety becoming obsolete to $1 - \varphi = 15\%$. This number is an intermediate value within the range of values of Kitao (2008) and Bilbiie, Ghironi and Melitz (2012) and is close to the 13.6% probability of Quadrini (2000) that self-employed business families quit business

[8] We base our target on two series by the Bureau of Economic Analysis: Until 1997, it published a series of intermediate inputs to gross output based on the Standard Industrial Classification (SIC) system before it was discontinued. Thereafter, it switched to the North American Industry Classification System (NAICS). For 1947–1997 (SIC), the ratio of intermediate inputs to gross output for all industries averages 0.45. For 1997–2025 (NAICS), the average is 0.44. Weighting each period by the number of annual observations yields a long-run weighted average of 0.45, which we use as our target.

[9] We view this counting of entrepreneurial income towards labor incomes as a short cut to modeling the innovation sector as using labor inputs instead of goods. Because of free entry, we get (close to) full dissipation of monopoly quasi rents as entry costs. In a model with labor as the only source of ideas and full dissipation of rents, we would obtain the total labor share of the economy as the labor share in production plus the share of monopoly quasi rents.

Table 1 Calibration Details (Annual Frequency)

Parameter	Value	Description	Source / Target
Households			
β	0.966	Discount factor	$(K_t + B_t)/Y_t = 3.275$
γ	2.00	Inverse Frisch	Chetty et al. (2011)
λ	5.86	Portfolio adj. prob. [in%]	Wealth Gini = 0.82
ω	0.91	Scale labor disutility	$N_t = 1.0$
ρ_h	0.95	Labor income persistence	Storesletten, Telmer and Yaron (2004)
σ_h	0.23	Labor income std.	$\text{Var}_{it}(\log(h_{it}n_{it})) = 0.84$
Firms			
ϵ	3.8	Substitution elasticity	Bilbiie, Ghironi and Melitz (2012)
ν	0.6	Share interm. inputs	$X_t/Z_t = 0.45$
δ	0.07	Depreciation rate	Bayer, Born and Luetticke (2023)
Government			
B_t/Y_t	0.4	Debt-to-GDP ratio	Bayer, Born and Luetticke (2023)
τ^L, τ^K, τ^π	25.7	Tax rate [in%]	$G_t/Y_t = 0.2$
Knowledge production			
φ	0.85	Prob. keeping equity	Quadrini (2000)
ρ	0.50	R&D externality	Kung and Schmid (2015)
χ	0.47	Inverse cost shifter R&D	$g_t = 2\%$

Note: All parameters in the table are calibrated to an annual frequency. Probabilities represent the likelihood within a single annual period.

activity. In our model, this number also pins down product life. The calibration implies an expected duration of a new product/a variety of 6.6 years. This is within the range of the expected product life estimated from (scanner) micro data in Adam and Weber (2023) and Argente, Lee and Moreira (2024) and the value used in Bilbiie, Ghironi and Melitz (2019). For the calibration of the knowledge production function we can rely on estimates from the representative households literature. Kung and Schmid (2015), Bianchi, Kung and Morales (2019) and Gaillard and Wangner (2022) obtain estimates of $\rho \approx 0.5$. Finally, we set $\chi = 0.47$ to target an annual growth rate of 2% (as in Moll, Rachel and Restrepo (2022)) along the balanced growth path.

While the model is calibrated to match certain data points, it closely replicates untargeted moments of the wealth distribution and returns on private equity. Table 2 demonstrates the fit to untargeted distributional moments. The model closely aligns with top wealth and income shares, though it marginally overstates the income Gini and the concentration of income at the top. Moreover, we overstate the number of households holding risky equity relative to the share of self-employed or business-owning households reported in Cagetti and De Nardi (2006).[10] Finally, the calibration yields an equilibrium expected after-tax return on equity of 8.5%, similar to measures in the

[10] Quadrini (2000) reports about 12% of US households as entrepreneurs, while Moskowitz and Vissing-Jørgensen (2002) report that a similar number of households report owning some private equity.

Table 2 Fit to moments of income and wealth distribution

	Wealth Shares		Income Shares		Gini Coeff.		Entrepr.
Source	Top 1%	Top 10%	Top 1%	Top 10%	Wealth	**Income**	Frac. $e > 0$
Model	0.27	0.69	0.16	0.49	0.82	0.59	0.20
Data	0.27	0.68	0.16	0.39	0.82	0.50	0.17

Note: All values are yearly. Data sources for the shares of wealth and income are the World Inequality Database from 1954-2019. The fraction of entrepreneurs is as reported in Cagetti and De Nardi (2006).

literature for the return on private equity (See Moskowitz and Vissing-Jørgensen, 2002; Fagereng et al., 2020; Bach, Calvet and Sodini, 2020).

5 Crowding in and out in the long-run

With this representation of the U.S. economy, we revisit the question posed by Aiyagari and McGrattan (1998) regarding the optimal level of government debt. To illustrate the quantitative significance of various mechanisms, we explore several distinct cases. First, we determine the welfare-maximizing level of debt within the complete model. We then examine the roles tax distortions and endogenous growth play in shaping our results. Finally, we also consider transition dynamics.

We use the utilitarian welfare function as the welfare measure for all experiments

$$W^* = \sum \mathbb{W}(a, e, h) d\Theta(a, e, h). \tag{41}$$

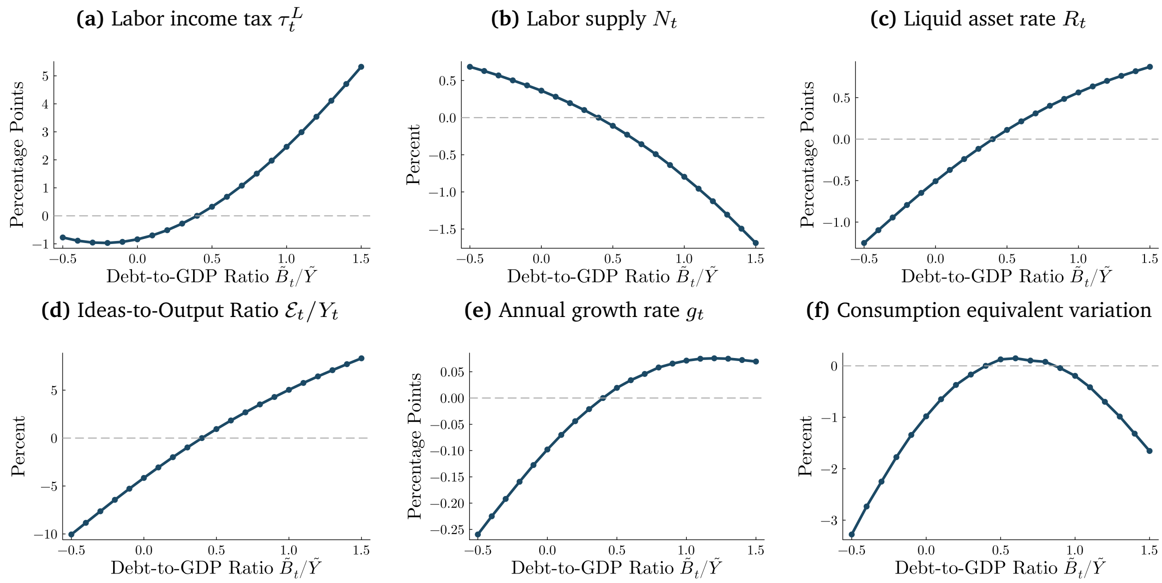
$W(a, e, h)$ denotes the continuation value from the household's problem, while Θ represents the stationary distribution. We use the continuation value rather than the value functions because the participation constraint in asset markets necessitates weighting the obtained value functions.[11] We utilize the consumption equivalent variation for logarithmic utility to compare different welfare levels

$$CE(\tilde{B}_t) = \exp\left((1-\beta)\left(W^*(\tilde{B}_t) - W_0^*\right)\right) - 1, \tag{42}$$

where W_0^* denotes welfare at the baseline level of government debt, and $W^*(\tilde{B}_t)$ denotes welfare at the debt level $\tilde{B}_t$. This measure represents the maximum fraction of consumption that the average household would be willing to forgo to remain on the baseline balanced growth path.

[11] We do not compute the value function directly, as we solve the stationary version of the model. Instead, we use the detrended value function and subsequently add the long-run growth rate according to equation (28). Appendix C illustrates the calculation.

Figure 2 Varying government debt and adjusting labor income tax residually



Note: The figure illustrates the values of variables along different balanced growth paths for different debt-to-GDP ratios $\tilde{B}_t/\tilde{Y}_t$. The x-axis refers to the ratio of debt-to yearly GDP such that the 1 refers to 100% of debt-to-GDP. Changes in the labor income tax, the liquid asset rate, the growth rate, and consumption equivalent variation are given in percentage points, while the other variables are illustrated in percent change.

5.1 The optimal long-run level of public debt

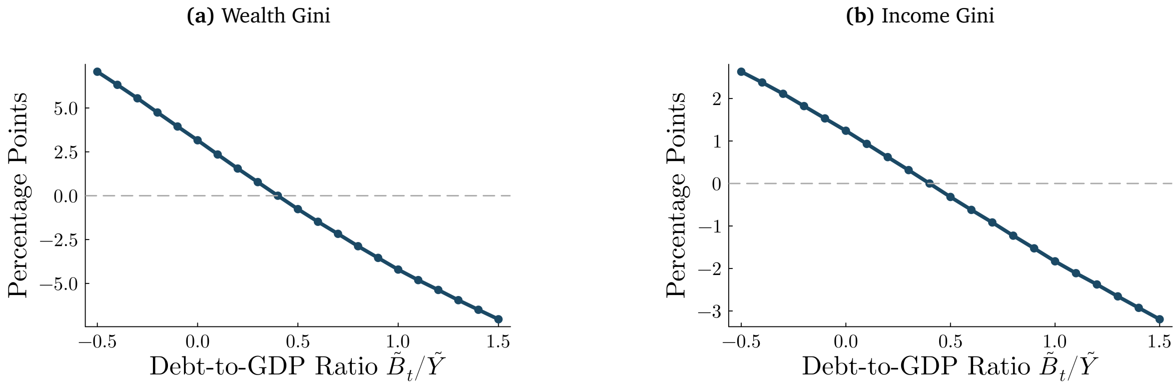
We now examine the welfare effects of changing the long-term government debt-to-GDP ratio in our baseline model. In this analysis, we assume the government balances the budget by adjusting the labor tax rate. The changes in selected variables as a result of this policy experiment are shown in Figure 2.

The higher the debt-to-output ratio, the higher labor income taxes have to be, see Figure 2 (a). As a result, after tax wages fall and households supply less labor, see Figure 2 (b). With an increased bond supply, interest rates rise to incentivize households to save more, see Figure 2 (c). In line with this change in the interest rate, the capital intensity falls (not displayed). However, the knowledge intensity, the varieties-to-output ratio, of the economy increases. Initially, this higher knowledge intensity goes hand in hand with higher growth, but eventually it is mostly driven by a crowding out of other factors. The growth rate of the economy peaks around a 120% debt-to-output ratio, see Figure 2 (e).[12] Yet, the welfare maximizing debt-to-output ratio is substantially smaller. At a debt-to-GDP ratio of 55%, consumption equivalent variation peaks at a value of 0.17%, with a growth rate 2.7 basis points higher than in the baseline balanced growth path. Once the debt-to-output ratio surpasses 55%, the required R&D investments outweigh the gains in growth, and the average consumption equivalent variation declines, see Figure 2 (f).

The fact that employment falls and the capital intensity declines follows the standard

[12] Note that we abstract from other effects of higher levels of government debt, such as rollover risk.

Figure 3 Changes of Inequality for different debt-to-GDP ratios



Note: Percentage points change in the wealth and income Gini compared to the baseline balanced growth path for different debt-to-GDP ratios.

intuition as in Aiyagari and McGrattan (1998). Higher public debt reduces the marginal value of liquidity and thus requires an increase in the interest rate for households to hold it. At the same time, the government needs to raise higher surpluses to service the larger amount of debt (given that we calibrated to a positive real-interest-rate-growth difference along the balanced growth path). This requires more distortionary labor taxes. Both effects of the higher debt lead directly and indirectly through the capital-labor complementarity to a lower employment of both factors.

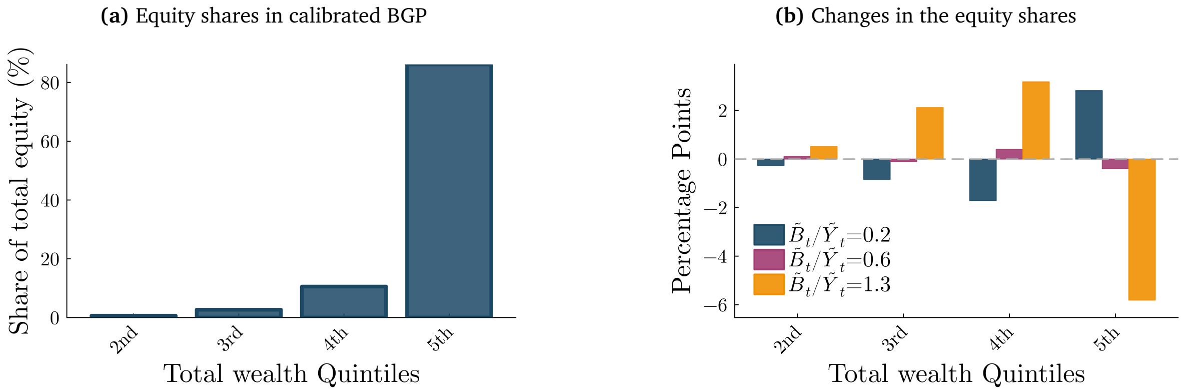
The fact that the knowledge intensity and, thereby, growth goes up shows that (the provision of) safe liquid assets and risky investments are complements. First, as the government issues more public debt, average household wealth goes up (by construction). The increase in liquid asset holdings is strongest for relatively poor households. As a result, wealth and income inequality decline when public debt increases. See Figure 3 for the wealth and income Gini as measures for wealth and income inequality.

This has consequences for investments in risky, illiquid assets. Figure 4 shows that as households are better insured and can better smooth their income in the event of a successful investment, more middle-class households invest in risky assets. Figure 4 (a) shows that, in line with the data, most entrepreneurial activity is concentrated among the very rich in our baseline.[13] Figure 4 (b) shows the changes in the equity shares compared to the baseline for different levels of government debt. When government debt increases (decreases), the middle class increases (decreases) its share of equity investment, while the share of the top quintile decreases (increases). In short: Higher debt and real rates make the innovative private equity market more inclusive.

The intuition behind this result rests on two key factors. First, increasing government debt reduces wealth inequality by enabling poorer households to accumulate wealth through higher interest rates. As their wealth grows, households are better

[13] The share of equity holdings in the first quintile is zero, such that we do not plot it. The next quintiles have shares of 0.64%, 2.67%, 10.54%, and 86.51%.

Figure 4 Relative investment contributions along wealth deciles



Note: Figure 4a shows the share of equity holdings by households per wealth quintile in the calibrated balanced growth path. We do not report the first quintile because their equity share is zero and does not change across debt-to-GDP ratios. Figure 4b shows the changes in the equity shares of households as the debt-to-GDP ratio changes.

insured against labor income risk and also invest more in risky projects. Second, higher interest rates on safe assets make risky, high-return assets more attractive in absolute terms, following the logic of the three-period model presented in section 2. Households with risky equity experience temporary high-income streams and save a large portion to smooth consumption. By raising the real interest rate, higher government debt improves their ability to transfer profit income across periods.

The change in welfare when changing debt masks a significant heterogeneity. One might be tempted to conclude from the decrease in wealth inequality (see Figure 3) that it should be particularly the poor who gain from an increase in debt. Panel A of table 3 shows that this is not the case. The table reports the average consumption equivalent variation of households in different wealth groups. That is, instead of using the average welfare across all households as in Equation 41, we calculate the consumption equivalent variation using (42) using the value functions

$$\hat{W}^* = \sum \mathbb{I}_{a+qe \in \text{wealth group}} \mathbb{W}(a, e, h) d\Theta(a, e, h),$$

hence, we compare welfare changes of households staying within a wealth group.[14]

Panel A of table 3 shows that welfare effects are heterogeneous. While the bottom nine deciles of the wealth distribution lose from higher debt, the top decile, and especially the top 1%, benefits from it. This distributional pattern emerges although, up until a debt-to-GDP ratio of 55%, the average welfare in the economy increases, such that our welfare measure Equation 42 increases. A welfare maximizing planner thus ex-ante would choose to increase the debt-to-GDP ratio to this point (not taking the transition into account). If households would vote about the long-run debt-to-GDP ratio they would like to live in, the majority of the households would vote against this

[14] Note that the average consumption equivalent variation reported for individual groups does not add up to the overall consumption equivalent measure due to Jensen's inequality.

Table 3 Average CEV and Income Shares for different Wealth Groups

Panel A: Consumption Equivalent Variation (%)				
$\tilde{B}_t/\tilde{Y}_t$	**Bottom 50%**	**Next 40%**	**Top 10%**	**Top 1%**
-0.40	8.7	8.0	2.9	-5.8
0.55	-1.7	-1.4	0.0	1.2
1.30	-9.9	-6.8	0.7	7.0

Panel B: Shares of Income Type in Total Income (%)				
	Bottom 50%	**Next 40%**	**Top 10%**	**Top 1%**
Labor	95.6	62.2	10.5	1.5
Asset	2.0	14.6	29.3	35.5
Equity	2.4	23.1	60.2	63.0

Note: Panel A reports the average consumption equivalent variation of different household groups for different debt-to-GDP ratios $\tilde{B}_t/\tilde{Y}_t$, compared to consumption of the respective groups in the baseline ($\tilde{B}_t/\tilde{Y}_t = 0.40$). Panel B reports income shares at the baseline balanced growth path. All figures in percent.

utilitarian welfare maximizing long-run optimal debt-to-GDP ratio.

Panel B of table 3 helps explain these findings by illustrating the shares of different income sources in the total income of the considered household groups. Households in lower wealth groups derive a significant portion of their income from labor, whereas households in the top decile receive much of their income from assets and equity. Since higher debt levels reduce labor income after taxes because taxes increase and capital intensity declines, an increase in debt harms the main income source of these households. An increase in debt also increases asset income and makes equity investments more attractive. Thus, it redistributes income to the wealthy. Of course, the effect on growth “raises all ships equally.”

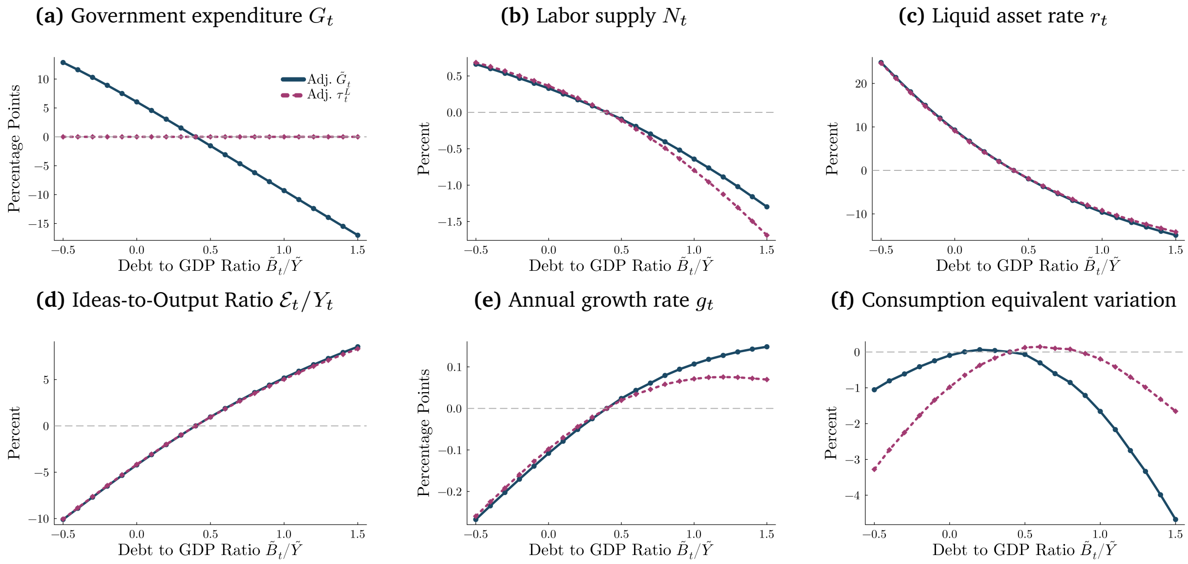
5.2 Dissecting the mechanism

After having illustrated the baseline result, we inspect the mechanism in modified environments. First, we illustrate how the results change if we adjust government expenditure instead of distortionary labor income taxes. Thereafter, we investigate the role of endogenous growth itself for the optimal debt-to-GDP ratio.

5.2.1 Adjusting government expenditures

To highlight the importance of tax distortions for the optimal level of debt and the crowding in of growth, we consider a version of our model, where the government adjusts government expenditures instead of taxes. To have meaningful welfare results, however, we replace the assumption of wasteful government expenditures with the assumption that households value government expenditures G_t. Here, we augment

Figure 5 Varying government debt and adjusting government expenditure residually



Note: The figure illustrates the values of variables along different balanced growth paths for different debt-to-GDP ratios $\tilde{B}_t/\tilde{Y}_t$ when government expenditure is adjusted to satisfy the government budget constraint (35). The x-axis refers to the ratio of debt-to yearly GDP such that the 1 refers to 100% of debt-to-GDP. Changes in the labor income tax, the liquid asset rate, the growth rate, and consumption equivalent variation are given in percentage points, while the other variables are illustrated in percent change.

each household's felicity function by a term $\zeta \log(G_t)$ and choose the weight ζ such that a modified Samuelson condition holds in our baseline balanced growth path.[15] Figure 5 illustrates the results of changing the debt-to-GDP ratio when financing it by adjusting government expenditure with the blue, dotted line. The results from our baseline exercise are illustrated through the red line with diamonds.

As debt rises, the government reduces expenditures to meet its budget constraint. As in the baseline case, the capital intensity of the economy decreases (not shown), which reduces wages. While as a result labor supply, and output still decrease, the decline is less severe than in the baseline case. The growth rate increases without a hump, unlike its evolution when adjusting the labor tax rate. Consumption equivalent variation follows an inverse U-shape similar to the baseline but rotated to the left around its origin, with a maximum at a debt-to-GDP ratio of 25% with consumption equivalent variation of 0.08%.

As in the baseline case, rising debt-to-GDP crowds out capital and labor, but the effect is less pronounced. This is because, rather than increasing labor taxes, the government cuts expenditures, which helps maintain a higher after-tax real wage, stimulating labor supply. While the income effect from higher after-tax wages reduces the crowding out of capital, it does so to a lesser extent than it mitigates the labor supply reduction. As a result, output declines less than in the baseline. When debt increases without the distortion from labor taxes, households invest more in risky assets than in the baseline

[15] We modify the Samuelson condition in the sense that there is no welfare gain if all households switch to a new balanced growth path with marginally more of the public good financed by an increase in public debt. We derive the modified Samuelson condition in Appendix D.1.

scenario, leading to a higher growth rate in comparison. Furthermore, households continue to invest in risky assets. This contrasts with the baseline scenario where risky investment and growth are crowded in only until a debt-to-GDP ratio of 120% due to the intensifying crowding out of labor and capital. Consumption equivalent variation exhibits a decline with an increase in the debt-to-GDP ratio as government expenditure, which households value, declines. This makes consumption equivalent variation peak at a lower debt-to-GDP ratio. Concretely, households prefer to have higher government expenditure and are willing to accept a lower long-run growth rate.

In summary, without the distortionary impact of labor taxes, the crowding out of capital and labor is reduced. Moreover, without the counteracting distortionary effect of labor taxation, growth does not exhibit a hump-shape as the debt-to-GDP ratio increases. As a result, crowding in is amplified relative to the baseline case, while households value higher government expenditure more than higher growth such that the optimal debt-to-GDP ratio is reduced below the baseline ratio.

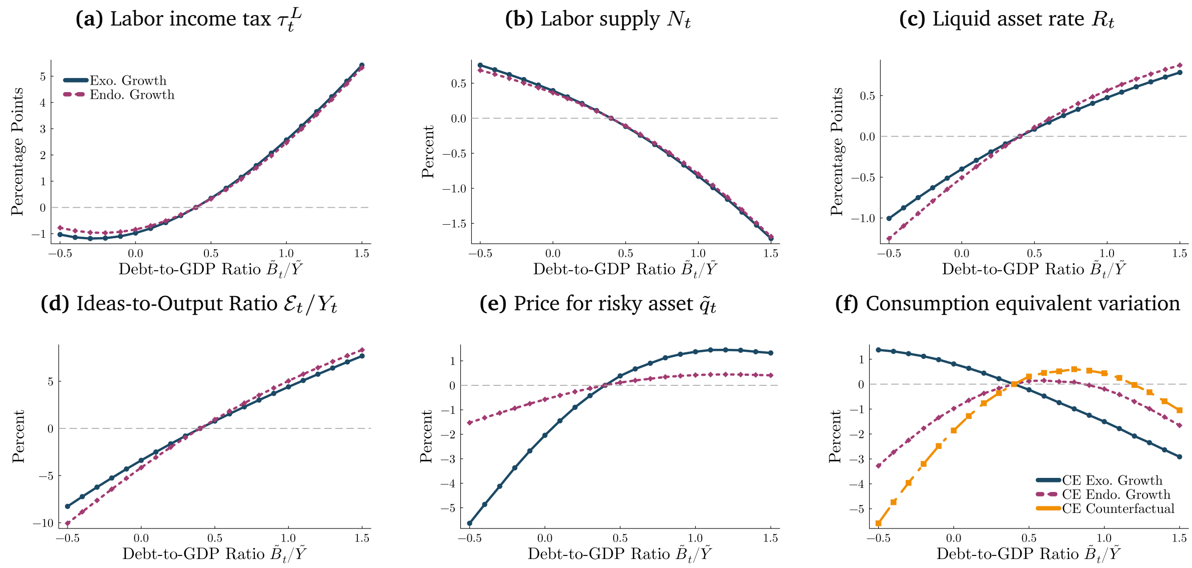
5.2.2 Exogenous growth

To highlight the importance of the endogenous growth effect for the optimal level of debt, we choose $\rho \approx 0$, which implies that the price of equities is almost infinitely elastic to changes in the quantity of equities demanded. Hence, a fixed number of new varieties is added to the economy every period, keeping the growth rate quasi exogenous. Varieties are traded as before, so no other element of the model changes. Figure 6 illustrates the results with exogenous growth in blue and compares it to the baseline results in red.

As the debt-to-GDP ratio rises, labor income taxes increase, and labor supply declines sharply, similar to the baseline case. However, capital decreases less than in the baseline (not shown), resulting in a smaller increase in the liquid asset rate. The price of risky investment rises more than in the baseline to clear the equity market without the growth rate moving. Consumption equivalent variation decreases monotonically with the debt-to-GDP ratio and differs significantly from the baseline, with the optimal debt-to-GDP ratio being negative, as in the exogenous growth framework of Röhrs and Winter (2015).

The crowding out of labor follows the same logic as in the baseline case. However, the difference in capital supply arises from the fixed growth rate. Growth affects the household's budget constraint by increasing the price of future asset holdings. To maintain the same amount of assets in efficiency units tomorrow, households must invest more today. Consequently, higher growth leads to higher investment costs, intensifying capital crowding out. With a fixed growth rate, this effect is absent, leading to less capital crowding out compared to the baseline scenario with endogenous growth. As a

Figure 6 Varying government debt and adjusting labor income tax with fixed growth



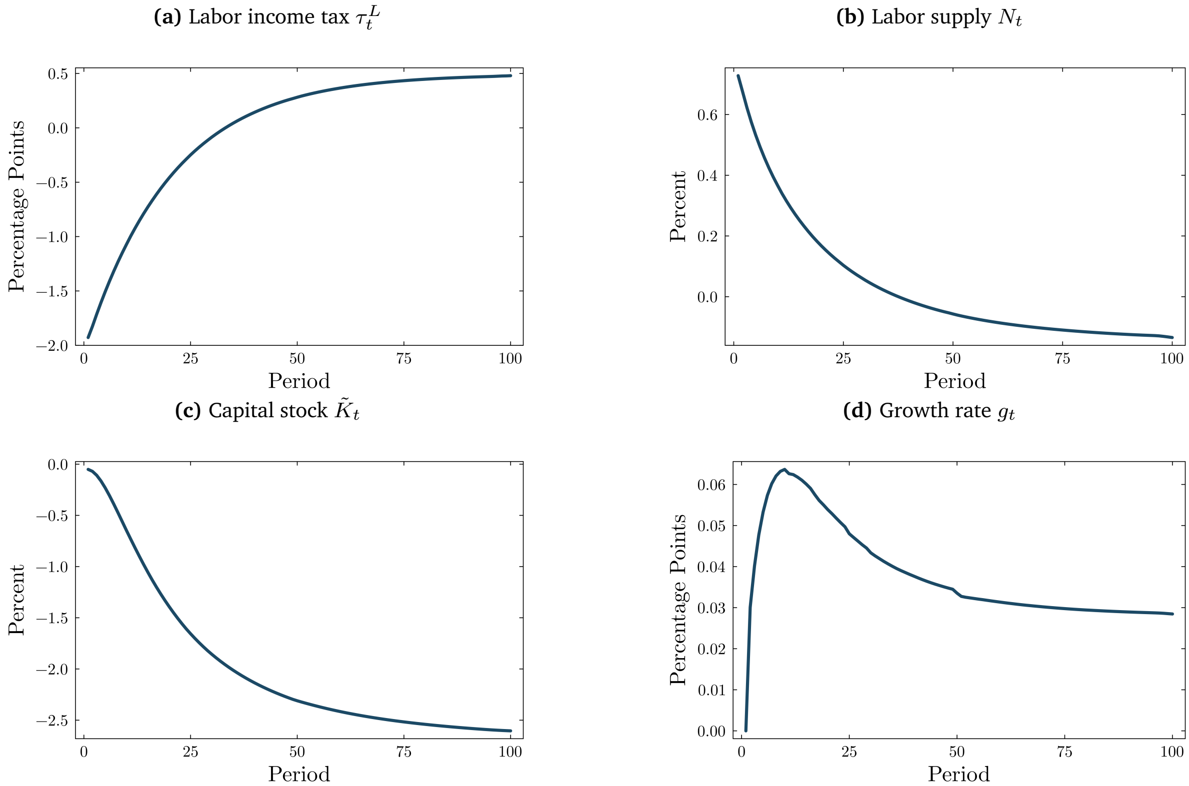
Note: The figure illustrates in blue the values of variables along different balanced growth paths for different debt-to-GDP ratios $\tilde{B}_t/\tilde{Y}_t$ with an exogenous growth rate. The x-axis refers to the ratio of debt-to yearly GDP such that the 1 refers to 100% of debt-to-GDP. Changes in the labor income tax, the liquid asset rate, the growth rate, and consumption equivalent variation are given in percentage points, while the other variables are illustrated in percent change. The red line illustrates the baseline experiment's results with endogenous growth. The yellow line in panel 6f also pictures the welfare of the exogenous growth economy when we readjust utility by the growth rate from the endogenous growth baseline.

result, the increase in the liquid asset rate is smaller, despite a similar decrease in labor supply. To clear the market for risky assets, households need to be disincentivized from investing, which requires a higher asset price. Since crowding out is lower and output drops less than in the baseline case, households have more resources available for investment. Therefore, the price of risky investments must increase more than in the baseline scenario to absorb the additional investment demand that households exhibit under a fixed growth rate.

The differential welfare effects are a result of the fixed growth assumption. Panel 6f illustrates consumption equivalent variation for the exogenous growth, the baseline case, and a counterfactual in yellow with squares. The counterfactual calculates welfare using the consumption allocation from the exogenous growth balanced growth path but adds the welfare contribution that households would obtain, when they would have the (endogenous) growth rate from the baseline case. Therefore, the difference between the blue and the yellow line illustrates the contribution of the endogenous change in the growth rate to welfare in the different scenarios. This comparison shows that the differences in welfare between these scenarios arise largely from the altered growth assumption. The counterfactual consumption equivalent variation is even higher for high government debt since the crowding out of capital is muted in the exogenous growth case, positively impacting consumption levels.

To sum up, with exogenous growth, the crowding out of capital and labor is reduced. Due to the absence of crowding in of growth, the welfare effects turn sign and the

Figure 7 Nonlinear perfect foresight transition to the optimal long-run debt level



Note: The figure illustrates the values of variables along a nonlinear perfect foresight transition to a higher debt-to-GDP ratio over 20 years. A period in the graphs is one year. The y-axis represent either percentage points differences or percentage changes compared to the baseline balanced growth path from which the transition starts.

optimal debt-to-GDP ratio is lower than in the baseline case with only small welfare increases.

5.3 Transition to the optimal balanced growth path

Finally, we analyze the welfare effects of the transition to the optimal debt-to-GDP level from our baseline analysis. To do so, we compute a nonlinear perfect foresight transition of the economy for a fixed path of government debt. Concretely, we let the debt level converge geometrically over 20 years to the new balanced growth path with a higher debt-to-GDP ratio. We then solve for the dynamic equilibria along this transition.

Figure 7 illustrates the results of this nonlinear transition. The government uses the funds collected from higher public debt to decrease the labor income tax initially by more than 2 percentage points. This results in a higher after-tax wage which incentivizes households to increase their labor supply by initially 0.7%. The increase in public debt crowds out the capital stock, such that households smoothly decrease their holdings until they hold a 2.8% lower capital stock. Finally, the growth rate of the economy overshoots the new long-run growth rate in the transition and then approaches the new growth rate from above.

Given these values for prices and quantities of state variables, we can compute the value functions of households and evaluate their welfare facing this transition. Expressed in consumption equivalent variation, households' welfare increases by 0.6%, with all households benefitting in utility terms. Hence, all households would vote in favor of this transition. This contrasts with the welfare results we obtain for the balanced growth paths the economy converges to. During the transition households experience higher long-run growth, an elevated short-run growth rate and higher consumption. Since initially labor income taxes are reduced, poor households obtain a transfer from rich households, which in turn will benefit in the long-run from higher interest rate income. In total, poor households favor the transition because they get compensated for the new long-run debt-to-GDP ratio, which they consider suboptimal. In Appendix E we show that this also holds for nonlinear transitions to economies with even higher debt-to-GDP ratios than in the baseline.

Hence, even though a majority of households would not choose to live on the higher-debt balanced growth path itself, the transition to it is welfare-improving for all of them. It might even be beneficial to transition into a balanced growth path with a lower long-run growth rate compared to the baseline case, as the transition provides welfare gains.

6 Conclusion

We develop a heterogeneous agent model with portfolio choice and endogenous growth. In our model, households face a portfolio decision between a risk-free asset, which offers insurance against idiosyncratic risk, and a risky asset that not only provides high returns but also contributes to economic growth. We show in this class of models that an increase in government debt crowds in growth and welfare. In a toy model, starting from a low value of debt, an increase in the latter raises the value of a successful risky investment for households, thereby crowding in investment as well as aggregate growth and increasing aggregate welfare. In a quantitative model calibrated to U.S. time series data, we revisit the question of the optimal level of public debt. Our findings confirm the intuition from the toy model. For small increases in government debt, the increased value from risky investment crowds in growth and welfare. For higher levels of government debt, classical crowding out effects on capital and labor outweigh crowding in effects.

When revisiting the question of the optimal level of public debt in the quantitative environment, our new channel results in a higher socially optimal level of public debt compared to previous studies. These welfare effects are distributed unequally, as wealthy households benefit from the higher long-run debt-to-GDP ratio. Dissecting the mechanism shows that the endogenous growth component is the main driver of the welfare result, whereas distortionary taxation is key for the hump-shaped growth pattern. While a majority of households would not vote for the higher-debt balanced growth path, the transition to it is welfare-improving for all households, as lower initial labor taxes and higher short-run growth more than compensate them along the path.

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Online Appendix

A Appendix: Derivations of the Toy model

The following subsection complements the result of section 2 in the main text. We fully solve the model assuming log utility. This allows us to derive the full allocation and welfare results analytically. We start by solving the model depending on the debt level. First, we solve the model with a low debt level such that only one household can save in the second period. Thereafter, we solve the model, with government debt high enough that both households can save in the second period. Based on the solution of the model, we then can calculate the critical debt level for which equity is crowded in by higher debt.

Description of the toy model

Household side: This section derives the solution to the household problem illustrated in section 2. Given prices, the households face the following maximization problem:

$$
\begin{aligned}
V(\omega) = &\max_{b_2, e, b_{3,H}, b_{3,L}} \ln(\omega - \tau_1 - b_2 - e) \\
&+ \varphi\Big[\ln(\tilde{\pi} e + R_1 b_2 + w_2 - \tau_2 - b_{3,H}) + \beta \ln(w_3 + R_2 b_{3,H} - \tau_3)\Big] \\
&+ (1-\varphi)\Big[\ln(R_1 b_2 + w_2 - \tau_2 - b_{3,L}) + \beta \ln(w_3 + R_2 b_{3,L} - \tau_3)\Big] \\
\text{s.t. } & b_2 \geq 0, b_{3,H} \geq 0, b_{3,L} \geq 0, e \geq 0,
\end{aligned}
$$

To solve the model, we proceed recursively, by characterizing the optimal savings decision in period two. Since period three marks the end of the lifetime, households optimally consume all their available resources.

Households enter period two with risk-free asset holdings b_2 and risky asset holdings e. We assume no heterogeneity in period two, such that households only differ in period two due to their realization of investment risk. We separate households depending on their realizations into groups $i \in \{H, L\}$, where $i = H$ denotes households with successful investments, while $i = L$ denotes households with unsuccessful investments. The successful households have $\tilde{\pi}_1 = \tilde{\pi}$, while the unsuccessful households have $\tilde{\pi}_2 = 0$. The asset positions b_2 and e, as well as the household group i are state variables for the household's decision problem. The optimal behavior of households in period two is characterized as a solution to the following problem:

$$V(b_2, e, i) = \max_{b_{3,i}} \ln(\tilde{\pi}_i e + R_1 b_2 + w_2 - \tau_2 - b_{3,i}) + \beta \ln(w_3 + R_2 b_{3,i} - \tau_3)$$
$$\text{s.t. } b_{3,i} \geq 0,$$

where the household takes assets b_2 and e, as well as the realization π_i as given. The first-order condition of the problem is a standard Euler equation with Lagrange multiplier ξ that is associated with the borrowing constraint.

$$\frac{1}{\pi_i e + R_1 b_2 + w_2 - \tau_2 - b_{3,i}} = \frac{\beta R_2}{w_3 + R_2 b_{3,i} - \tau_3} + \xi(b_2, e, i) \tag{43}$$

where $\xi(b_2, e, i)$ denotes that the Lagrange multiplier depends on the individual states.

In the first period, households face a portfolio problem between the risk-free asset b_2 and the risky asset e. The solution to the portfolio problem is determined by the set of Euler equations

$$\frac{1}{\omega - b_2 - e - \tau_1} = \frac{\varphi \tilde{\pi}}{\tilde{\pi} e + R_1 b_2 + w_2 - \tau_2 - b_{3,H}} + \mu(\omega), \tag{44}$$

$$\frac{1}{\omega - b_2 - e - \tau_1} = R_1 \left[\frac{\varphi}{\tilde{\pi} e + R_1 b_2 + w_2 - \tau_2 - b_{3,H}} + \frac{1 - \varphi}{R_1 b_2 + w_2 - \tau_2 - b_{3,L}} \right] + \kappa(\omega), \tag{45}$$

where $\mu(\omega)$ and $\kappa(\omega)$ denote the Lagrange multipliers associated with the nonnegativity constraints of the risky and the risk-free asset. Having defined the optimality conditions for the households, we continue with the other agents in the economy.

Production side: The representative firm produces in periods two and three according to the production function

$$Y = \mathcal{E} N.$$

In period two, the firm rents labor at the wage $w_2 = \phi Y$ and generates profits $\pi = (1 - \phi) Y$ that are paid out to households that hold the risky asset in period two. The total profits are distributed among all ideas $\mathcal{E}$, such that the individual return that households obtain is $\tilde{\pi} = \frac{\pi}{\mathcal{E}}$. The payout structure of the firm emerges, for example, from a two-level production structure with intermediate goods producers that produce under monopolistic competition and a perfectly competitive final goods producer. In this structure, ϕ denotes the inverse of the markup that the intermediate goods firms charge. In period three, the payout structure changes in the sense that the firm only has to pay labor such that $w_3 = Y$. This payout structure emerges if the intermediate goods producers lose their ability to exploit the monopoly. Hence, we assume that in the long run (which period three represents), product markets become perfectly competitive.

Government: In period one, the government issues government debt $\mathcal{B}$ that matures in period two to finance transfers $-\tau_1$ to the household in period one. Therefore

$$-\tau_1 = \mathcal{B}.$$

In period two, the government has to repay its outstanding government debt plus the interest on it $R_1\mathcal{B}$ through lump-sum taxes τ_2 and issuing new debt $\mathcal{B}$. Therefore the government budget constraint is

$$\tau_2 = (R_1 - 1)\mathcal{B}.$$

In period three the government repays its outstanding government debt and the interest payment associated with it $R_2\mathcal{B}$ through lump-sum taxes τ_3. The budget constraint in period three, therefore, is

$$\tau_3 = R_2\mathcal{B}.$$

Market clearing: Market clearing requires that goods markets, labor markets, risk-free and risky asset markets clear. Asset market clearing requires

$$\mathcal{B} = \int_0^1 b_2 di = b_2, \quad \text{and} \quad \mathcal{B} = \varphi b_{3,H} + (1-\varphi) b_{3,L}$$

in the first period and the second period for the risk-free asset, and

$$\mathcal{E} = \varphi \int_0^1 e_i di = \varphi e$$

for the risky asset in the first and second period. Market clearing in the labor market requires $N = 1$ in both periods since households inelastically supply one unit of labor. Finally, goods market clearing requires

$$\begin{aligned} \omega &= \int (c_i + e_i) di \\ Y &= \varphi c_{2,H} + (1-\varphi) c_{2,L}, \\ \text{and } Y &= \varphi c_{3,H} + (1-\varphi) c_{3,L}. \end{aligned}$$

Equilibrium: An equilibrium in this economy consists of policy functions $\{b_2^*, e^*, b_{3,H}^*, b_{3,L}^*\}$, pricing functions $w_2, \tilde{\pi}, w_3$, aggregate labor and equity functions $\{\mathcal{E}, L\}$ such that the following statements hold:

1. Given prices, the policy functions solve the household planning problem.

2. The labor, the bond, the equity, and the goods market clear, the return on equity and the wage rate are determined competitively (i.e. by the firm's problem), while the interest rate on bonds is determined via bond market clearing.

After the description of the model and its equilibrium, we turn to the solution of it.

Solution of the toy model

Solving the model requires solving the household problem while imposing market-clearing conditions in the asset markets and substituting the prices from the firm side. All assumptions allow us to solve the model in closed form. However, the model features a solution that depends on the level of government debt. If government debt is scarce, the unsuccessful households are constrained in period two such that only the successful households are willing to save. On the contrary, if government debt is abundant both households can save in period two, effectively completing asset markets.

Case with scarce liquidity

To start, we are interested in the case where liquidity is scarce such that only the successful households are willing to save in government debt.[16] In the following, we solve the model by determining the optimal behavior of the unconstrained households. Lemma 1 summarizes the solution of the household problem in period two

Lemma 1. *Assume that only the successful households can save in the risk-free asset in period two. In equilibrium, consumption in period two is*

$$c_{2,H} = \tilde{\pi}e + w_2 - \left(\frac{1-\varphi}{\varphi}\right)\mathcal{B} \tag{46}$$

$$c_{2,L} = w_2 + \mathcal{B}. \tag{47}$$

Consumption in period three is

$$c_{3,H} = w_3 + \left(\frac{1-\varphi}{\varphi}\right)R_2\mathcal{B} \tag{48}$$

$$c_{3,L} = w_3 - R_2\mathcal{B}. \tag{49}$$

Finally, the real interest rate between periods two and three is

$$R_2 = \frac{w_3}{\beta\left(\tilde{\pi}e + w_2\right) - (1+\beta)(\frac{1-\varphi}{\varphi})\mathcal{B}}. \tag{50}$$

[16] Corollary 1 states the conditions that are necessary for the successful households to be the only households on the Euler equation.

Proof. To obtain the consumption in the two periods, we use that consumption in periods two and three is $c_{2,i} = \pi_i e + R_1 b_2 + w_2 - \tau_2 - b_{3,i}$ and $c_{3,i} = w_3 + R_2 b_{3,i} - \tau_3$.

The assumption that the successful households are the only ones able to save implies that the other household group is borrowing constrained $b_{3,L} = 0$ and faces autarky. Market clearing in period two then implies that successful households have to save $b_{3,H} = \frac{\mathcal{B}}{\varphi}$. We then obtain the consumption functions of households, by using the budget constraints of the government in periods two and three to substitute the tax rates out of the budget constraints of the households. The real interest rate is then determined via the Euler equation (43) of the successful household. □

Lemma 1 summarizes the solution of the consumption saving problem in the second period in the case with scarce liquidity, i.e. that only the successful household saves. Equations (46), (47), and (48) show that albeit being scarce, in equilibrium, government debt helps to smooth the consumption of households. Unsuccessful households can use government debt to increase consumption in period two, effectively borrowing against income in period three in which they need to pay taxes. The high-households use government debt in period two to transfer some resources from the successful state in period two to the lower-income state in period three, thereby smoothing consumption. Effectively, while being scarce, government debt reduces the dispersion of marginal utility of consumption in period two.

Finally, we can characterize the real interest rate analytically in equation (50). In the special case, the real interest rate R_2 increases with the amount of government debt. Moreover, the expression for the real interest rate allows us to characterize a condition on parameters such that high households are indeed the only household that saves between periods two and three. We derive the condition that implies $\xi(b_2, e, L) > 0$ in the Euler equation for the unsuccessful households.

Corollary 1. *If*

$$\frac{\beta}{1+\beta}\varphi\tilde{\pi}e > \mathcal{B}, \tag{51}$$

then unsuccessful households are constrained and successful households are the only households that are not constrained in period two.

Proof. We use the consumption functions in periods two and three and substitute them into the inequality

$$\frac{1}{w_2 + \mathcal{B}} > \frac{\beta R_2}{w_3 - R_2\mathcal{B}},$$

that determines that the unsuccessful-households are constrained. Substituting for the real interest rate (50) and rearranging terms yields the upper bound (51). □

Corollary 1 provides an upper bound on government debt for the successful households to be the only households to save. Intuitively the condition states that the amount of government debt is not allowed to be higher than the income difference between successful households and unsuccessful households in period two. If this condition were violated, unsuccessful households would accept a lower interest rate to save than successful households, such that they would become the new marginal savers. The condition does not yet acknowledge that the rents the firm pays depend on the amount invested in the risky asset $\mathcal{E}$. We restate the condition considering this below. To be able to do so, however, we require an expression for the amount of investment in the risky asset.

After having specified the optimal behavior of the household in the second period, we turn to the optimal behavior of households in the first period. To solve the household decision problem we use the Euler equations (44) and (45) together with market clearing conditions to obtain an analytical solution for the choices of b_2 and e. Since we know the exact expressions for consumption in the second period, and since all households are ex-ante identical in the economy, we can exactly characterize the equilibrium policies. Lemma 2 characterizes the portfolio choices in equilibrium.

Lemma 2. *Assume that government debt is scarce in the sense of Corollary 1. Then in period one, households invest*

$$b_2 = \mathcal{B} \tag{52}$$

into the risk-free asset and

$$e^* = \frac{\varphi(1-\phi)\omega + \frac{(1-\varphi)}{\varphi}\mathcal{B}}{(1+\varphi-\phi)} \tag{53}$$

into the risky asset. Consumption in period one is

$$c_1 = \omega - e^*. \tag{54}$$

The risk-free interest rate R between periods one and two is defined as

$$R_1 = \frac{\frac{1}{c_1}}{\frac{\varphi}{c_{2,H}} + \frac{(1-\varphi)}{c_{2,L}}}, \tag{55}$$

where $c_{2,i}$ denotes the consumption of household group i in period two.

Proof. Since all households are identical in the first period, market clearing requires each of the households to hold the total amount of government debt such that $b_2 = \mathcal{B}$, deriving equation (52). Since the government budget constraint in the first period requires $-\tau_1 = \mathcal{B}$, this implies that transfers and government debt cancel each other in the first-period budget constraint. This implies that consumption in period one is $c_1 = \omega - e$, deriving (54).

To obtain the optimal saving in the risky asset e, we substitute consumption for the successful household in period two into the Euler equation for the risky asset (44). Solving the equation for e and substituting in yields equation (53). With all households being identical in the first period, this implies $e = \mathcal{E}/\varphi$. Finally, the risk-free interest rate follows from the Euler equation for the risk-free asset and from market clearing. We abstract from stating the exact expression here but only state the dependence of the interest rate on the consumption policies in period two from Lemma 1, and consumption in period one that is a function of optimal savings in e defined in equation (53). □

Lemma 2 provides us with the last policy functions to characterize the solution to the household problem in the case of scarce government debt. It is worth mentioning that the optimal investment of households in the risky asset e is an increasing function in government debt $\mathcal{B}$. The reason for this is that in period two, consumption decreases in the amount of government debt since the successful household uses scarce debt to smooth consumption between periods two and three. Therefore, if the government increases the amount of government debt, all households will increase their risky investment, since they anticipate that they will have fewer resources in period two if they become successful households.

Having specified the optimal portfolio choice, we can make our condition in Corollary 1 more concrete and relate the upper bound to model parameters. The following Corollary 2 summarizes the condition as function of the model parameters.

Corollary 2. *If*

$$\mathcal{B} < \frac{\beta\varphi^2(1-\phi)^2\omega}{(1-\phi)+\varphi(1+\beta(2-\phi))} \equiv \mathcal{B}^*, \tag{56}$$

successful households are the only households willing to save in the second period. The threshold $\mathcal{B}^*$ is always positive.

Proof. The upper bound follows from substituting the optimal portfolio choice (53) into equation (51) and rearranging. $\mathcal{B}^* > 0$ follows from the parameter constraints $0 < \phi < 1$ and $0 < \varphi < 1$. □

Corollary 2 provides an upper bound on government debt such that households that experience a positive asset income realization have sufficiently higher income than households with low income. Equation (56) defines the bound for proposition 1 and proposition 2 in the main text.

Case with abundant liquidity

Next, we derive the solution to the household problem if government debt is abundant in the sense that Corollary 2 does not hold and both households save in period two.

This impacts consumption smoothing and the portfolio choice in period one. To obtain the solution of the problem in this case, we repeat the steps from above.

Lemma 3. *Assume that Corollary 2 does not hold. Then in period two, each household group i saves*

$$b_{3,i} = \mathcal{B} + \frac{\beta}{1+\beta}(\mathbb{I}_{i=H}\tilde{\pi}e + w_2 - w_3), \tag{57}$$

where $\mathbb{I}_{i=H}$ is the indicator for when a households has H type, yielding consumption of the two household groups to be

$$c_{2,H} = \frac{\tilde{\pi}e + w_2}{1+\beta} + \frac{\beta}{1+\beta}w_3 \tag{58}$$

$$c_{2,L} = \frac{w_2}{1+\beta} + \frac{\beta}{1+\beta}w_3. \tag{59}$$

Finally, the real interest rate between periods two and three is

$$R_2 = \frac{1}{\beta} \tag{60}$$

Proof. If both groups are on the Euler equation (43), then optimal consumption and savings is related by

$$\frac{1}{c_{2,i}} = \frac{\beta R_2}{c_{3,i}}, \tag{61}$$

where we can substitute in the budget constraints for periods two and three. Solving the Euler equation for the optimal savings policy yields

$$b_{3,i} = \frac{\beta}{1+\beta}(\mathbb{I}_{i=H}\tilde{\pi}e + w_2 + R_1 b_2 - \tau_2) - \frac{w_3 - \tau_3}{(1+\beta)R_2}. \tag{62}$$

We can impose that all households are identical in the first period such that $b_2 = \mathcal{B}$, as well as the budget constraints of the government to obtain that the policy function for savings is

$$b_{3,i} = \mathcal{B} + \frac{\beta}{1+\beta}\left(\mathbb{I}_{i=H}\tilde{\pi}e + w_2\right) - \frac{w_3}{(1+\beta)R_2}. \tag{63}$$

Using the market clearing condition of the asset market $\varphi b_{3,H} + (1-\varphi)b_{3,L} = \mathcal{B}$, as well as the expressions for prices w_2, w_3 and $\tilde{\pi}$, as well as the fact that output Y is constant over time yield the expression for the real interest rate (60). We can use this expression to simplify (63) to obtain the expression for savings (57). We obtain the expressions for consumption by substituting the optimal amount of savings into the budget constraints of the households. □

Lemma 3 states the solution to the second period's household problem. Since government debt is abundant, both household groups are on the Euler equation and save in the risk-free asset. This enables them to smooth their consumption between periods two and three making consumption of both households independent of the level of government debt. This has the important implication for the portfolio problem that investment in the risky asset becomes independent from the amount of government debt, as well. We state this result formally in Lemma 4

Lemma 4. *Assume that Corollary 2 does not hold. Then in period one, households in equilibrium invest*

$$b_2 = \mathcal{B} \tag{64}$$

into the risk-free asset and

$$e^* = \frac{\phi(1-\varphi)(1+\beta)}{(1-\phi)+\varphi(1+2\beta-\phi\beta)}\omega \tag{65}$$

into the risky asset. The risk-free interest rate R between periods one and two is defined as in expression (55).

Proof. Identical to the former argument, since all households are identical in the first period, market clearing requires each of the households to hold the total amount of government debt such that $b_2 = \mathcal{B}$, deriving equation (64). To obtain the optimal saving in the risky asset e, we substitute consumption for the successful household in period two into the Euler equation for the risky asset (44). Solving the equation for e yields equation (65). With all households being identical in the first period, this implies $\mathcal{E} = \varphi e$. □

Lemma 4 finalizes the solution of the model version with abundant government debt. If Corollary 2 does not hold, then there exists sufficient liquidity in the economy for both household groups to save in the second period and turn the model effectively Ricardian. This disables the formerly active channel that government debt crowds in risky investments. The reason for this is that government debt does not facilitate consumption smoothing for the successful household, since household decisions are independent of the level of government debt.

Derivation of welfare results

Having solved the model in its two versions, we can now derive the welfare effect that an increase in government debt $\mathcal{B}$ has on the economy. To do so, we follow a similar approach as Dávila et al. (2012), however within a three-period model. We take the derivative of the value function with respect to $\mathcal{B}$, simplify using the optimality conditions and market clearing, and then establish the sign using a continuity argument around $\mathcal{B} = 0$.

Proposition 3. *There exists a threshold $\hat{\mathcal{B}}^* > 0$ below which households' lifetime welfare is strictly increasing in public debt: $\frac{\partial V}{\partial \mathcal{B}} > 0$ for all $\mathcal{B} \in (0, \hat{\mathcal{B}}^*)$.*

Proof. **Step 1: Simplifying the derivative.**

Differentiating the value function $V(\omega)$ with respect to $\mathcal{B}$ yields

$$\begin{aligned}\frac{\partial V(\omega)}{\partial \mathcal{B}} &= u'(c_1)\left[-\frac{\partial e}{\partial \mathcal{B}} - \frac{\partial \tau_1}{\partial \mathcal{B}} - \frac{\partial b_2}{\partial \mathcal{B}}\right] \\ &+ \varphi\Bigg[u'(c_{2,H})\left(\tilde{\pi}\frac{\partial e}{\partial \mathcal{B}} + \frac{\partial \tilde{\pi}}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}}e + \frac{\partial R_1}{\partial \mathcal{B}}a + R_1\frac{\partial b_2}{\partial \mathcal{B}} + \frac{\partial w_2}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} - \frac{\partial \tau_2}{\partial \mathcal{B}} - \frac{\partial b_{3,H}}{\partial \mathcal{B}}\right) \\ &+ \beta u'(c_{3,H})\left(\frac{\partial w_3}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} - \frac{\partial \tau_3}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}}b_{3,H} + R_2\frac{\partial b_{3,H}}{\partial \mathcal{B}}\right)\Bigg] \\ &+ (1-\varphi)\Bigg[u'(c_{2,L})\left(\frac{\partial R_1}{\partial \mathcal{B}}a + R_1\frac{\partial b_2}{\partial \mathcal{B}} + \frac{\partial w_2}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} - \frac{\partial \tau_2}{\partial \mathcal{B}} - \frac{\partial b_{3,L}}{\partial \mathcal{B}}\right) \\ &+ \beta u'(c_{3,L})\left(\frac{\partial w_3}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} - \frac{\partial \tau_3}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}}b_{3,L} + R_2\frac{\partial b_{3,L}}{\partial \mathcal{B}}\right)\Bigg]. \end{aligned} \tag{66}$$

Several groups of terms drop out. The terms involving $\partial e/\partial \mathcal{B}$ and $\partial b_2/\partial \mathcal{B}$ cancel by the period-1 Euler equations for e and b_2 respectively. The terms involving $\partial b_{3,H}/\partial \mathcal{B}$ cancel by the period-2 Euler equation of the H-type household. The terms involving $\partial b_{3,L}/\partial \mathcal{B}$ vanish because the L-type is constrained in the scarce-liquidity case: both $b_{3,L}$ and its derivative are zero. Imposing the government budget constraints ($\frac{\partial(-\tau_1)}{\partial \mathcal{B}} = 1$, $\frac{\partial \tau_2}{\partial \mathcal{B}} = \frac{\partial R_1}{\partial \mathcal{B}}\mathcal{B} + (R_1 - 1)$, $\frac{\partial \tau_3}{\partial \mathcal{B}} = \frac{\partial R_2}{\partial \mathcal{B}}\mathcal{B} + R_2$) and market clearing ($b_2 = \mathcal{B}$), the remaining terms simplify to

$$\begin{aligned}\frac{\partial V(\omega)}{\partial \mathcal{B}} &= \varphi\Bigg[u'(c_{2,H})\left(\frac{\partial \tilde{\pi}}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}}e + \frac{\partial w_2}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}}\right) + \beta u'(c_{3,H})\left(\frac{\partial w_3}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}}(b_{3,H} - \mathcal{B})\right)\Bigg] \\ &+ (1-\varphi)\Bigg[u'(c_{2,L})\frac{\partial w_2}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \beta u'(c_{3,L})\left(\frac{\partial w_3}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}}(b_{3,L} - \mathcal{B})\right) + \xi(\mathcal{B}, \mathcal{E}, 2)\Bigg], \end{aligned} \tag{67}$$

where $\xi(\mathcal{B}, \mathcal{E}, 2) \geq 0$ is the Lagrange multiplier on the L-type's period-2 borrowing constraint.

Step 2: Evaluating at $\mathcal{B} = 0$.

We now assess the sign of each term in (67) at $\mathcal{B} = 0$. We evaluate the relevant derivatives at $\mathcal{B} = 0$ in turn.

Factor price derivatives. From the production structure with $\tilde{\pi} = (1-\phi)$ and $w_2 = \phi\mathcal{E}$, $w_3 = \mathcal{E}$:

$$\frac{\partial \tilde{\pi}}{\partial \mathcal{E}} = 0, \qquad \frac{\partial w_2}{\partial \mathcal{E}} = \phi > 0, \qquad \frac{\partial w_3}{\partial \mathcal{E}} = 1 > 0. \tag{68}$$

Derivative of investment. From Lemma 2, equilibrium investment is $e^* = [\varphi(1-\phi)\omega + (1-\varphi)\mathcal{B}/\varphi]/(1+\varphi-\phi)$. Therefore

$$\frac{\partial \mathcal{E}}{\partial \mathcal{B}} = \varphi \frac{\partial e^*}{\partial \mathcal{B}} = \frac{1-\varphi}{1+\varphi-\phi} > 0, \tag{69}$$

which is strictly positive since $\varphi < 1$ and $1+\varphi-\phi > 1-\phi > 0$.

Sign of $\partial R_2/\partial \mathcal{B}$ at $\mathcal{B} = 0$. From $R_2 = w_3/[\beta(\tilde{\pi}e + w_2) - (1+\beta)\frac{1-\varphi}{\varphi}\mathcal{B}]$, differentiating with respect to $\mathcal{B}$ and evaluating at $\mathcal{B} = 0$ gives

$$\left.\frac{\partial R_2}{\partial \mathcal{B}}\right|_{\mathcal{B}=0} = \left.\frac{(1+\beta)\frac{1-\varphi}{\varphi}\, w_3}{[\beta(\tilde{\pi}e + w_2)]^2}\right|_{\mathcal{B}=0} > 0. \tag{70}$$

Higher government debt raises R_2 near $\mathcal{B} = 0$: the denominator of R_2 shrinks as $\mathcal{B}$ grows.

Sign of $\xi(\mathcal{B}, \mathcal{E}, 2)$ at $\mathcal{B} = 0$. The L-type constraint binds at $\mathcal{B} = 0$ if and only if its unconstrained optimum requires borrowing, i.e. if

$$\frac{1}{c_{2,L}} > \frac{\beta R_2}{c_{3,L}}. \tag{71}$$

At $\mathcal{B} = 0$, the allocations from Lemma 1 reduce to $c_{2,L} = w_2 = \phi\mathcal{E}$ and $c_{3,L} = w_3 = \mathcal{E}$. Since $\phi < 1$ we have $c_{2,L} < c_{3,L}$, so $1/c_{2,L} > 1/c_{3,L}$. The interest rate at $\mathcal{B} = 0$ is $R_2|_{\mathcal{B}=0} = \varphi/[\beta(1-\phi+\phi\varphi)] < 1/\beta$, so

$$\left.\frac{\beta R_2}{c_{3,L}}\right|_{\mathcal{B}=0} = \left.\frac{\varphi}{(1-\phi+\phi\varphi)\, c_{3,L}}\right|_{\mathcal{B}=0} = \frac{1}{(1-\phi+\phi\varphi)\,\mathcal{E}/\varphi} < \left.\frac{1}{c_{3,L}}\right|_{\mathcal{B}=0} < \left.\frac{1}{c_{2,L}}\right|_{\mathcal{B}=0}, \tag{72}$$

where the first inequality uses $1-\phi+\phi\varphi < 1$ since $\phi > 0$ and $\varphi < 1$. Condition (71) therefore holds strictly, so $\xi(\mathcal{B}, \mathcal{E}, 2) > 0$ at $\mathcal{B} = 0$. Economically, the L-type's income is lower in period 2 than in period 3 (it receives no monopoly rents), so it wishes to borrow against its future wage income but is prevented from doing so.

The two period-3 conditions at $\mathcal{B} = 0$. With $b_{3,H} - \mathcal{B} = b_{3,L} - \mathcal{B} = 0$ at $\mathcal{B} = 0$, the $\partial R_2/\partial \mathcal{B}$ terms in (67) drop out entirely:

$$\left.\left(\frac{\partial w_3}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}}(b_{3,H} - \mathcal{B})\right)\right|_{\mathcal{B}=0} = \frac{1-\varphi}{1+\varphi-\phi} > 0, \tag{73}$$

$$\left.\left(\frac{\partial w_3}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}}(b_{3,L} - \mathcal{B})\right)\right|_{\mathcal{B}=0} = \frac{1-\varphi}{1+\varphi-\phi} > 0. \tag{74}$$

Both expressions equal $\frac{1-\varphi}{1+\varphi-\phi} > 0$ since $\varphi < 1$ and $\phi < 1$.

Step 3: Sign of $\frac{\partial V}{\partial \mathcal{B}}$ at $\mathcal{B} = 0$.

Substituting all of the above into (67) at $\mathcal{B} = 0$, using $\partial\tilde{\pi}/\partial\mathcal{E} = 0$ and log utility

$u'(c) = 1/c$:

$$\left.\frac{\partial V(\omega)}{\partial \mathcal{B}}\right|_{\mathcal{B}=0} = \frac{1-\varphi}{1+\varphi-\phi}\left\{\varphi\left[\frac{\phi}{c_{2,H}} + \frac{\beta}{c_{3,H}}\right] + (1-\varphi)\left[\frac{\phi}{c_{2,L}} + \frac{\beta}{c_{3,L}}\right]\right\} + \xi(0, \mathcal{E}, 2). \quad (75)$$

To see how (75) arises, note that $\partial w_2/\partial \mathcal{E} \cdot \partial \mathcal{E}/\partial \mathcal{B} = \phi(1-\varphi)/(1+\varphi-\phi)$ and $\partial w_3/\partial \mathcal{E} \cdot \partial \mathcal{E}/\partial \mathcal{B} = (1-\varphi)/(1+\varphi-\phi)$. Factoring out $(1-\varphi)/(1+\varphi-\phi)$ and replacing $u'(c) = 1/c$ then yields (75) directly. Since all consumption levels are strictly positive at $\mathcal{B} = 0$, every term inside the braces is strictly positive, the prefactor $(1-\varphi)/(1+\varphi-\phi) > 0$, and $\xi(0, \mathcal{E}, 2) > 0$. Therefore

$$\left.\frac{\partial V(\omega)}{\partial \mathcal{B}}\right|_{\mathcal{B}=0} > 0. \quad (76)$$

Step 4: Continuity argument.

All quantities entering (67), the equilibrium allocations c_1, $c_{2,H}$, $c_{2,L}$, $c_{3,H}$, $c_{3,L}$, the interest rate R_2, and the derivatives $\partial \mathcal{E}/\partial \mathcal{B}$, $\partial R_2/\partial \mathcal{B}$, are continuous functions of $\mathcal{B}$ on the interval $[0, \mathcal{B}^*)$. Since (76) establishes strict positivity at $\mathcal{B} = 0$, by continuity there exists $\hat{\mathcal{B}}^* \in (0, \mathcal{B}^*)$ such that

$$\frac{\partial V(\omega)}{\partial \mathcal{B}} > 0 \quad \text{for all } \mathcal{B} \in (0, \hat{\mathcal{B}}^*). \quad (77)$$

This completes the proof. □

Similarly, we can derive

Corollary 3. *In the abundant-liquidity case, government debt has no effect on household welfare:* $\frac{\partial V}{\partial \mathcal{B}} = 0$.

Proof. The proof uses the simplified expression (67) derived in the proof of Proposition 3. In the abundant-liquidity case, all three channels through which debt affects welfare are inactive simultaneously.

Channel 1: Investment and wages. By Lemma 4, optimal investment e^* is independent of $\mathcal{B}$, so $\partial \mathcal{E}/\partial \mathcal{B} = 0$. The terms involving $\partial w_2/\partial \mathcal{E}$ and $\partial \tilde{\pi}/\partial \mathcal{E}$ in (67) therefore vanish, eliminating the wage and profit channels.

Channel 2: Interest rate. By Lemma 3, the real interest rate is $R_2 = 1/\beta$ irrespective of $\mathcal{B}$, so $\partial R_2/\partial \mathcal{B} = 0$. The terms involving $(b_{3,H} - \mathcal{B})$ and $(b_{3,L} - \mathcal{B})$ in (67) therefore vanish as well, eliminating the portfolio-smoothing channel.

Channel 3: Borrowing constraint. In the abundant-liquidity case, both household groups are on their Euler equation by assumption, so $\xi(\mathcal{B}, \mathcal{E}, 2) = 0$.

With all three channels inactive, every term in (67) equals zero, giving $\partial V/\partial \mathcal{B} = 0$. □

Corollary 3 establishes that Ricardian equivalence holds exactly in the abundant-liquidity case. The intuition is straightforward: once liquidity is sufficient for both household groups to access the bond market, households undo any change in public debt one-for-one through their private savings decisions, leaving consumption allocations, factor prices, and hence welfare unchanged. Together with Proposition 3, this implies that the welfare gains from government debt are entirely concentrated in the scarce-liquidity region $\mathcal{B} \in (0, \hat{\mathcal{B}}^*)$.

Corollary 4. *The welfare threshold satisfies $\hat{\mathcal{B}}^* \geq \mathcal{B}^*$, so that $\frac{\partial V}{\partial \mathcal{B}} > 0$ holds throughout the entire scarce-liquidity region $\mathcal{B} \in (0, \mathcal{B}^*)$, if and only if*

$$\phi \;\geq\; \phi^*(\varphi, \beta) \;\equiv\; \frac{3\beta\varphi + \varphi + 2 - \sqrt{\varphi(4\beta^3\varphi + 5\beta^2\varphi + 4\beta^2 + 2\beta\varphi + 4\beta + \varphi)}}{2(\beta\varphi + 1)}. \tag{78}$$

For the baseline calibration ($\varphi = 0.85$, $\beta = 0.966$) this threshold is $\phi^* \approx 0.41$, which is comfortably below the implied labor share $\phi \approx 0.72$.

Proof. **Step 1: A boundary value of R_2.**

We first establish a useful property of the scarce-liquidity equilibrium at $\mathcal{B} = \mathcal{B}^*$. At the boundary, Corollary 2 holds with equality, so

$$\mathcal{B}^* = \frac{\beta}{1+\beta}\,\varphi\,\tilde{\pi}\,e^*(\mathcal{B}^*) \quad\Longrightarrow\quad (1+\beta)\frac{1-\varphi}{\varphi}\,\mathcal{B}^* = \beta(1-\varphi)\,\tilde{\pi}\,e^*(\mathcal{B}^*). \tag{79}$$

Substituting into the denominator of R_2 (equation 50):

$$\begin{aligned} \beta(\tilde{\pi}e^* + w_2^*) - (1+\beta)\frac{1-\varphi}{\varphi}\mathcal{B}^* &= \beta[\tilde{\pi}e^*(1-(1-\varphi)) + w_2^*] = \beta[\varphi\tilde{\pi}e^* + w_2^*] \\ &= \beta\varphi e^* = \beta w_3^*. \end{aligned} \tag{80}$$

Therefore $R_2(\mathcal{B}^*) = w_3^*/(\beta w_3^*) = 1/\beta$: the real interest rate equals the households' discount factor exactly at the scarce-liquidity boundary, the same value that prevails throughout the abundant-liquidity case. Economically, $\mathcal{B}^*$ is the debt level at which the L-type's borrowing constraint just stops binding, so the bond market begins to price the L-type's Euler equation, the same condition that holds throughout the abundant-liquidity case and that drives Corollary 3.

Step 2: Reducing to a single condition.

The derivative $\partial V/\partial \mathcal{B}$ in (67) remains strictly positive as long as all four bracketed terms are positive. As shown in the proof of Proposition 3, the H-type period-3 term and the two period-2 wage terms are always positive. The only term that can become negative is the L-type period-3 bracket, since $b_{3,L} - \mathcal{B} = -\mathcal{B} < 0$. The condition $\hat{\mathcal{B}}^* \geq \mathcal{B}^*$ therefore holds if and only if this bracket is non-negative at $\mathcal{B} = \mathcal{B}^*$:

$$\left.\frac{\partial w_3}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathcal{B}} + \frac{\partial R_2}{\partial \mathcal{B}}\,(b_{3,L} - \mathcal{B})\right|_{\mathcal{B}=\mathcal{B}^*} = \frac{\partial \mathcal{E}}{\partial \mathcal{B}} - \frac{\partial R_2}{\partial \mathcal{B}}\,\mathcal{B}^* \;\geq\; 0. \tag{81}$$

Substituting $\partial\mathcal{E}/\partial\mathcal{B} = (1-\varphi)/(1+\varphi-\phi)$ and the expression for $\partial R_2/\partial\mathcal{B}$ evaluated at $\mathcal{B} = \mathcal{B}^*$ (using $R_2(\mathcal{B}^*) = 1/\beta$ from Step 1 and $\mathcal{B}^*/w_3^* = \beta(1-\phi)/(1+\beta)$ from (79)), condition (81) becomes

$$\frac{1-\varphi}{1+\varphi-\phi} \geq \frac{(1-\varphi)(1-\phi)\big[\beta\varphi(2-\phi)+(1+\varphi-\phi)\big]}{\beta\varphi(1+\beta)(1+\varphi-\phi)}. \tag{82}$$

Multiplying both sides by the positive factor $\beta\varphi(1+\beta)(1+\varphi-\phi)/(1-\varphi)$ to clear denominators and rearranging yields the equivalent condition $P(\varphi,\phi,\beta) \geq 0$, where

$$P(\varphi,\phi,\beta) \equiv \beta^2\varphi + \beta\varphi\big(3\phi-\phi^2-1\big) - \phi^2 + \phi(\varphi+2) - \varphi - 1. \tag{83}$$

Step 3: Signing P and solving for the threshold.

P is a downward-opening quadratic in ϕ. Evaluating at the boundary values gives

$$P\big|_{\phi=0} = \varphi(\beta^2-\beta-1) - 1 < 0 \qquad \text{(for all } \beta\in(0,1)), \tag{84}$$

$$P\big|_{\phi=1} = \beta\varphi(\beta+1) > 0. \tag{85}$$

Since P is continuous and changes sign on $(0,1)$, there is a unique root $\phi^*(\varphi,\beta)\in(0,1)$ such that $P\geq 0$ if and only if $\phi\geq\phi^*$. Solving $P=0$ as a quadratic in ϕ and taking the root in $(0,1)$ yields the expression in (78). The condition $\hat{\mathcal{B}}^* \geq \mathcal{B}^*$ therefore holds if and only if $\phi\geq\phi^*(\varphi,\beta)$.

Step 4: Calibration check.

At the baseline values $\varphi = 0.85$ and $\beta = 0.966$, evaluating (78) gives $\phi^* \approx 0.41$. The implied labor share $\phi\approx 0.72$, computed from the calibrated production parameters $\epsilon = 3.8$ and $\nu = 0.6$ via $\phi = \epsilon(1-\nu)/[\epsilon(1-\nu)+\nu]$, exceeds this threshold, so $P>0$ and $\hat{\mathcal{B}}^* > \mathcal{B}^*$ at the calibration. Hence under the baseline parameterization, $\partial V/\partial\mathcal{B} > 0$ on the entire scarce-liquidity region $(0,\mathcal{B}^*)$, the welfare-improving range of public debt is not just a neighborhood of zero but coincides with the full scarce-liquidity regime. □

The economic content of (78) is intuitive. When ϕ is low, labor receives only a small share of output, so the wage channel through which higher debt raises household income is weak. At the same time, the L-type is still obliged to pay the rising $R_2\mathcal{B}$ in taxes in period three. If this interest-rate burden grows faster than the wage benefit, the L-type's period-3 welfare decreases in $\mathcal{B}$ before the scarce-liquidity region ends. A sufficiently large labor share $\phi\geq\phi^*(\varphi,\beta)$ ensures the wage channel always dominates.

B Appendix: Solution of the household's problem

Given the structure mentioned in the main text, we now characterize its solution. In the following, we skip all time indexes of value functions and refer to future realizations of variables with a prime. We can write the first-order conditions as functions of the shadow values. These conditions read

$$
\begin{aligned}
e_a^*: \qquad & q \times \frac{u(c_a^*(a,e,h), n_a^*(a,e,h))}{\partial c} &&= \beta \times \frac{\partial \mathbb{W}_{t+1}(a^a(a,e,h), e^a(a,e,h), h)}{\partial e'} && (86)\\
a_a^*: \qquad & \frac{u(c_a^*(a,e,h), n_a^*(a,e,h))}{\partial c} &&= \beta \times \frac{\partial \mathbb{W}_{t+1}(a^a(a,e,h), e^a(a,e,h), h)}{\partial a'} && (87)\\
a_n^*: \qquad & \frac{u(c_n^*(a,e,h), n_n^*(a,e,h))}{\partial c} &&= \beta \times \frac{\partial \mathbb{W}_{t+1}(a_n(a,e,h), e, h)}{\partial a^n(a,e,h)} && (88)\\
n_a^*: \qquad & -\frac{u(c_a^*(a,e,h), n_a^*(a,e,h))}{\partial n} &&= (1-\tau^L) w_t h_{it} \frac{u(c_a^*(a,e,h), n_a^*(a,e,h))}{\partial c} && (89)\\
n_n^*: \qquad & -\frac{u(c_n^*(a,e,h), n_n^*(a,e,h))}{\partial n} &&= (1-\tau^L) w_t h_{it} \frac{u(c_n^*(a,e,h), n_n^*(a,e,h))}{\partial c} && (90)
\end{aligned}
$$

Therefore, to solve the model numerically, we require expressions for the continuation values. In a first step, we require the Envelope conditions:

$$
\begin{aligned}
\frac{\partial V_a(a,e,h)}{\partial a} &= R(a, R^a) \times \frac{\partial u(c_a^*(a,e,h), n_a^*(a,e,h))}{\partial c} && (91)\\
\frac{\partial V_n(a,e,h)}{\partial a} &= R(a, R^a) \times \frac{\partial u(c_n^*(a,e,h), n_n^*(a,e,h))}{\partial c} && (92)\\
\frac{\partial V_a(a,e,h)}{\partial e} &= (q + \Pi) \times \frac{\partial u(c_a^*(a,e,h), n_a^*(a,e,h))}{\partial c} && (93)\\
\frac{\partial V_n(a,e,h)}{\partial e} &= (\Pi - gq) \times \frac{\partial u(c_n^*(a,e,h), n_n^*(a,e,h))}{\partial c}\\
&\quad + \beta \times \frac{\partial \mathbb{W}_{t+1}(a_n^*, e, h)}{\partial e} && (94)
\end{aligned}
$$

With these envelope conditions, we can derive the derivative of $\mathbb{W}$ with respect to the states a and e. These derivatives can be interpreted as the shadow value of the liquid asset and the equity asset for a household at the specified position of the state space.

$$
\begin{aligned}
\frac{\partial \mathbb{W}_{t+1}}{\partial a}(a,e,h) &= R(a, R^a) \times \Bigg\{ \varphi \mathbb{E}\left[\lambda \frac{\partial V_a(a,e,h)}{\partial a} + (1-\lambda)\frac{\partial V_n(a,e,h)}{\partial a}\right]\\
&\qquad + (1-\varphi)\mathbb{E}\left[\lambda \frac{\partial V_a(a,0,h)}{\partial a} + (1-\lambda)\frac{\partial V_a(a,0,h)}{\partial a}\right]\Bigg\} && (95)\\
\frac{\partial \mathbb{W}_{t+1}}{\partial e}(a,e,h) &= \varphi \mathbb{E}\left[\lambda \frac{\partial V_a(a,e,h)}{\partial e} + (1-\lambda)\frac{\partial V_n}{\partial e}(a,e,h)\right]. && (96)
\end{aligned}
$$

The shadow value for equity has a recursive structure due to its dependence on its own derivative via the term $\frac{\partial V_n}{\partial e}$.

Finally, note that if households are constrained, they still optimally supply labor. To determine the optimal labor supply, households solve the following static optimization problem:

$$\begin{aligned}
&\max_{n_{it}} \ln(c_{it}) - \omega \frac{n_{it}^{1+\gamma}}{1+\gamma} \\
&\text{s.t. } c_{it} = (1-\tau^L) w_t h_{it} n_{it} + T_{it} + (1+r) a_{it} + (\mathcal{I}_{\text{adj.}} q_t + \pi_t) e_{it} - \underline{a}
\end{aligned}$$

Solving the first-order condition of the constraint problem gives us the following expression for leisure

$$n_{it} = \left(\frac{(1-\tau^L) w_t h_{it}}{\tilde{c}_{it} \omega} \right)^{\frac{1}{\gamma}}, \tag{97}$$

that implicitly defines labor supply. We precompute this expression before solving the household problem. Given the optimality conditions we can characterize a solution algorithm for the individual problem.

C Appendix: Solution method

We use an algorithm similar to Aiyagari (1994), Aiyagari and McGrattan (1998) to compute the stationary equilibrium. The model will be solved by guessing the capital stock K_t, labor supply N_t, the growth rate of the economy g_t, and the tax rate on labor income τ_t^L. Given these guesses, we compute the policy functions via the endogenous grid method originally developed by Carroll (2006) and subsequently developed by Hintermaier and Koeniger (2010), and Bayer and Luetticke (2020) to a two-asset structure. The solution of the model with labor supply is based on the appendix of Auclert, Bardóczy and Rognlie (2023). We aggregate the economy via the histogram method of Young (2010). If aggregate supply matches aggregate demand in all markets, the algorithm has converged. The pseudo-code goes as follows:

1. Guess $\{K_t, N_t, g_t, \tau_t^L\}$
   (a) Compute the interest rate R_t, the wage rate w_t, and the price for new equity q_t.
   (b) Guess the policy functions $c_a^*, c_n^*, n_a^*, n_n^*$ and value functions V_a, and V_n. In the first iteration guess the shadow value $\frac{\partial \mathbb{W}_{t+1}(a,e,h)}{\partial e} = \mathbf{0}$. Precompute the optimal labor supply if households are at the budget constraint.
   (c) Use equation (88) to solve for an updated policy function for a_n^*, n_n^* and c_n^* using standard endogenous grid methods. We elaborate below in detail on how to solve for leisure.

(d) Combine equations (86) and (87) and find off-grid values for $\hat{a}(e,h)$ related to the exogenous grid of e'

$$0 = \frac{\partial \mathbb{W}_{t+1}(\hat{a}(e',h'),e',h)}{\partial e'} - \frac{\partial \mathbb{W}_{t+1}(\hat{a}(e',h'),e',h)}{\partial a'}. \tag{98}$$

(e) From equation (87) compute an update for the marginal utility of consumption today. Use the marginal utility to update c_a^* and n_a^*. Use the policy functions and the optimal choice of $\hat{a}'$ obtained in step 3 for a fixed grid of e_a^* to construct an endogenous grid. Interpolate the policy functions from the endogenous grid on the exogenous grid.

(f) Check whether the borrowing constraints are binding in any of the two cases. If they are binding, force households to allocate all their income into consumption and leisure according to eq. (97).

(g) Update the value for the shadow value of liquid assets $\frac{\partial \mathbb{W}_{t+1}(a,e,h)}{\partial a}$ with the new policy functions. Update the shadow value of equity $\frac{\partial \mathbb{W}_{t+1}(a,e,h)}{\partial e}$ with the new policy functions and the former guess for the shadow value of equity based on equation (96).

(h) Repeat steps b) to g) until convergence in all policy functions occurs.

2. Aggregate the economy up and check for market clearing given the guesses from step 1.

3. If market clearing is achieved, stop. If not, iterate on the guesses $\{K_t, N_t, g_t, \tau_t^L\}$.

To obtain the labor supply, note that labor is given as a function of the marginal utility of consumption. This relation holds for the adjustment case, as well as the non-adjustment case. We obtain the marginal utility defined on the endogenous grid from the Euler equation in both cases. We then calculate the policy function for labor supply on the endogenous grid. Given labor supply, we can construct the labor income that households obtain. With the labor income of households, we can compute the endogenous grid and interpolate back onto the exogenous grid.

Having found an equilibrium in the economy, we then proceed to compute the continuation value $\mathbb{W}$, which we use for welfare evaluations. Based on (28), the value functions can be written as

$$V^a(a,e,h) = u(c_a^*, n_a^*) + \ln(1+g) + \beta \mathbb{W}(a_a^*, e_a^*, h)$$
$$\text{and} \quad V^n(a,e,h) = u(c_n^*, n_n^*) + \ln(1+g) + \beta \mathbb{W}(a_n^*, e, h),$$

where we have assumed that $\mathcal{E}_{-1} = 1$ such that in this period $\mathcal{E} = 1+g$ along a balanced growth path. Consequently, we can decompose the value functions V^a and V^n into a stationary component and a growth component. Let $\tilde{V}^i(a,e,h)$ denote the stationary component. For log utility, each future period t contributes an additional $t \cdot \ln(1+g)$ to utility, so the growth component follows from the geometric sum $\sum_{t=0}^{\infty} \beta^t t = \beta/(1-\beta)^2$, giving

$$V^i(a,e,h) = \tilde{V}^i(a,e,h) + \frac{\beta}{(1-\beta)^2} \ln(1+g), \quad i \in \{a,n\}. \tag{99}$$

The stationary value functions $\tilde{V}^i$ are computed first via Bellman iteration using $\tilde{\beta}$, and the growth component is then added. The continuation value follows from (34).

D Appendix: Model variants

This subsection includes the description of the alternative model versions we solve with the government adjusting government expenditures to clear the budget and with an exogenous growth rate.

D.1 Adjusting Government debt

When adjusting government expenditure instead of the labor income tax, we augment the utility function (3.2.1) by a term $\zeta \ln(G_t)$ such that households obtain utility from government expenditure. This enables us to obtain non-trivial welfare results. Writing utility as a function of detrended variables

$$\mathbb{E}_0 \max_{\{\tilde{c}_{it}, n_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\ln(\tilde{c}_{it}) - \omega \frac{n_{it}^{1+\gamma}}{1+\gamma} + \zeta \ln(\tilde{G}_t) + (1+\zeta) \ln(\mathcal{E}_t) \right], \tag{100}$$

households obtain additional utility from detrended government expenditure and experience stronger utility gains from the number of varieties in the economy. The latter effect appears since government expenditure G_t grows by the growth rate of the economy, as does individual consumption c_{it}. Using the identical calculation as in Appendix III, we can decompose the value function when valuing government expenditure $V_G^i, i \in \{a,n\}$ into the component without government expenditure V^i and the component that is added through government expenditure:

$$V_G^i(a,e,h) = V(a,e,h) + \zeta \frac{\beta}{(1-\beta)^2} \ln(1+g) + \zeta \frac{\ln(\tilde{G}_t)}{1-\beta}, \tag{101}$$

where the last two terms follow due to the utility of government expenditure and infinite sums. This implies that the continuation value with utility from government expenditure is

$$\mathbb{W}_G(a,e,h) = \mathbb{W}(a,e,h) + \zeta\frac{\beta}{(1-\beta)^2}\ln(1+g) + \zeta\frac{\ln(\tilde{G}_t)}{1-\beta}, \tag{102}$$

For the welfare results to be meaningful, we derive a modified Samuelson condition. Samuelson (1954) shows that the optimal supply of a public good is determined by the condition that the marginal rate of substitution between the public good and a consumption good is equal to their relative price. While the original condition is derived in a static environment we extend the condition to a dynamic setting. We modify the Samuelson condition in the sense that on average there is no welfare gain if all households switch to a new balanced growth path with marginally more of the public good financed by an increase in public debt. To achieve this, we set the parameter ζ. We impose the condition

$$\frac{\partial \mathbb{W}_G(a,e,h)}{\partial B} = 0$$

and solve for ζ, which yields

$$\zeta = -(1-\beta)\left(\frac{\beta}{1-\beta}\frac{\partial g/\partial \tilde{B}}{(1+g)} + \frac{\partial \tilde{G}^*/\partial \tilde{B}}{\tilde{G}^*}\right)^{-1}\int \frac{\partial \mathbb{W}(b_i,e_i,h_i)}{\partial \tilde{B}}di. \tag{103}$$

Intuitively, ζ is set such that around the calibrated balanced growth path, the welfare effect that a small change in the amount of government debt has via changing the growth rate or detrended government expenditure is compensated on average. We approximate each of the derivatives numerically using a symmetric derivative for a small perturbation around the baseline debt level.

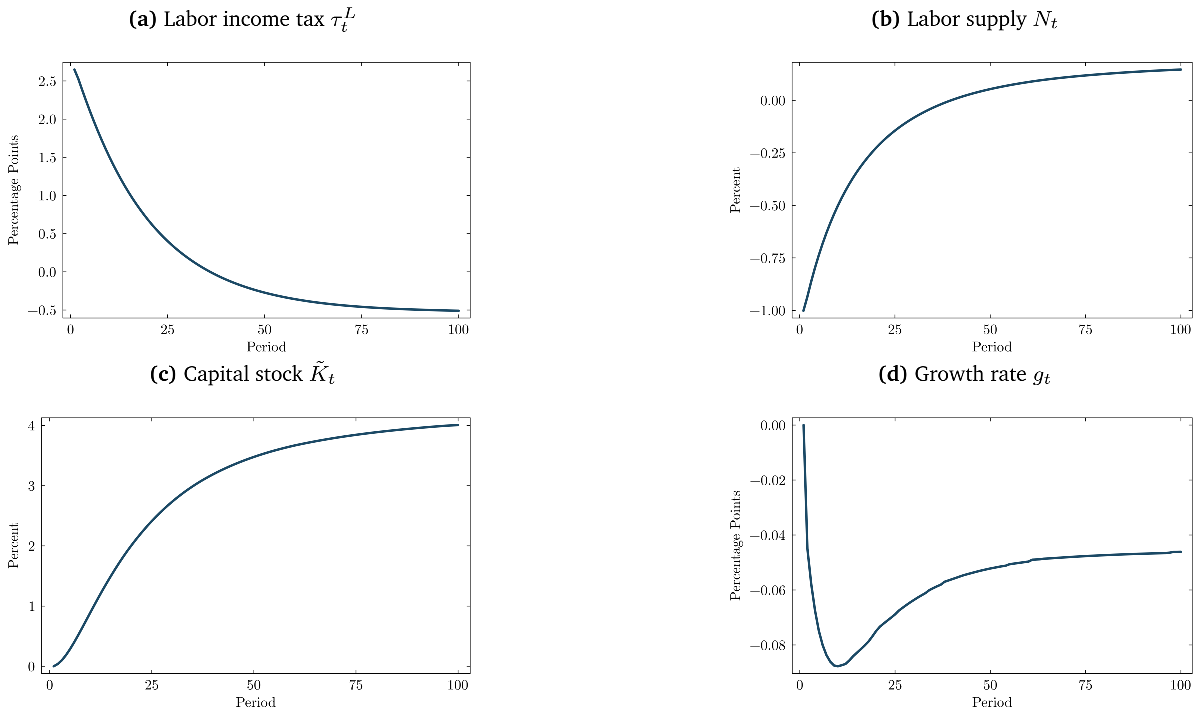
D.2 Fixed exogenous growth rate

We consider an alternative specification of the model by taking the limit $\rho \to 0$. In this regime, the price of equity q_t becomes highly sensitive to new varieties, such that even marginal changes in the number of new varieties $\tilde{\Delta}_t$ induce large movements in q_t. Consequently, the equity market clears primarily through adjustments in the asset price rather than through changes in the equilibrium growth rate.

E Additional Results for Nonlinear Transitions

This section provides additional plots for nonlinear transitions of the economy from the baseline balanced growth path to different long-run debt-to-GDP ratios. Figure E.1, Figure E.2, and Figure E.3 illustrate the nonlinear dynamics to debt-to-GDP ratios of 0.2, 0.6, and 1.2, respectively. All figures illustrate that the shape of the transition

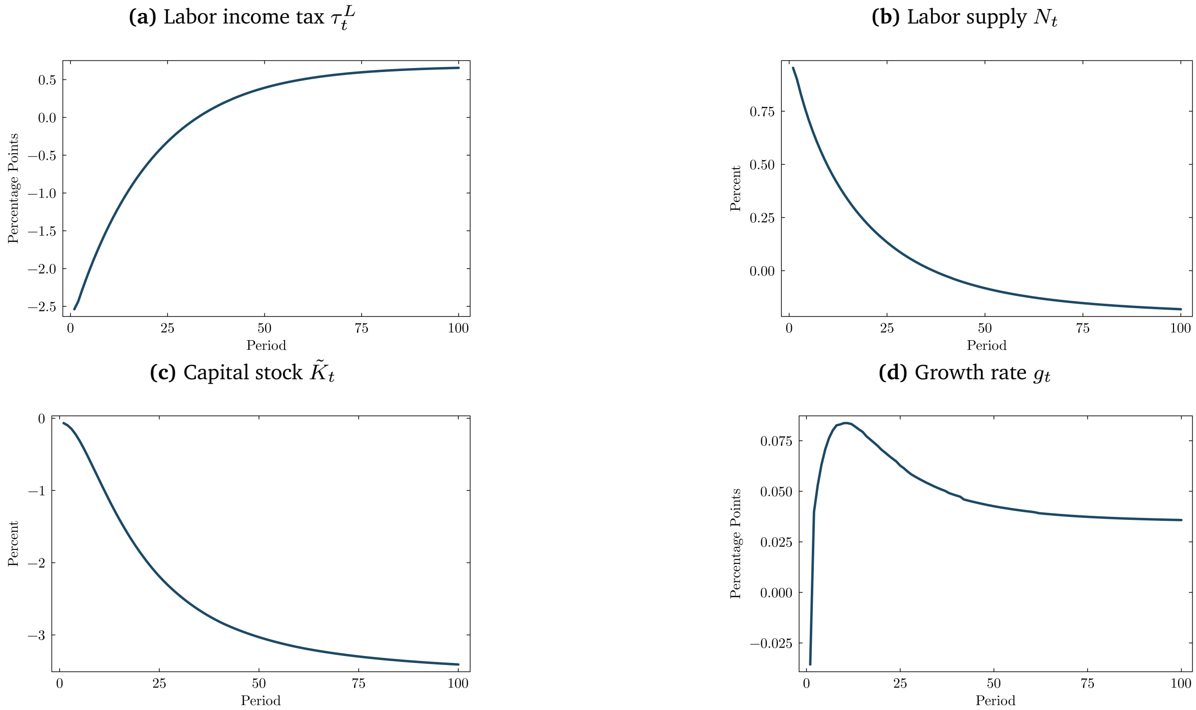
Figure E.1 Nonlinear perfect foresight transition to the debt-to-GDP ratio of 0.2



Note: The figure illustrates the values of variables along a nonlinear perfect foresight transition to a 20% lower debt-to-GDP ratio over 20 years. A period in the graphs is one year. The y-axis represent either percentage points differences or percentage changes compared to the baseline balanced growth path from which the transition starts.

remains identical, in the sense that balanced growth paths with higher debt-to-GDP ratios always feature initially lower labor tax rates, higher labor supply, a decrease in detrended capital, as well as overshooting of the growth rate. The transition to a lower debt-to-GDP ratio features the same patterns reversed.

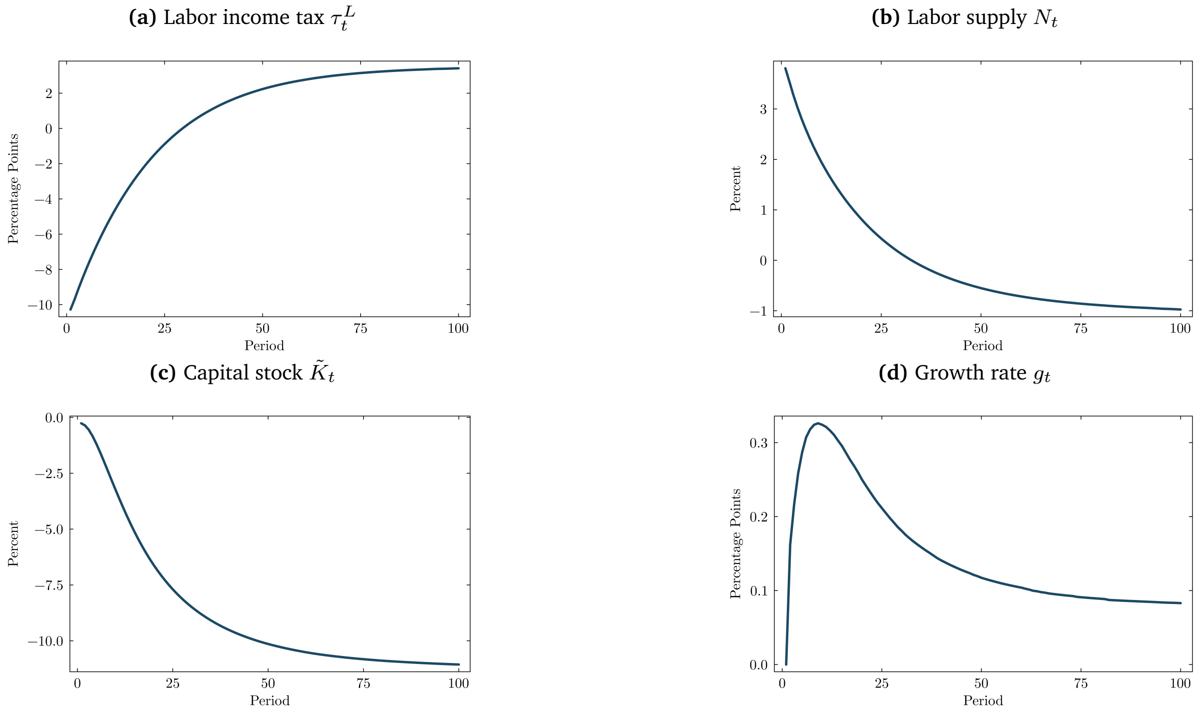
Figure E.2 Nonlinear perfect foresight transition to the debt-to-GDP ratio of 0.6



(a) Labor income tax τ_t^L **(b)** Labor supply N_t **(c)** Capital stock $\tilde{K}_t$ **(d)** Growth rate g_t

Note: The figure illustrates the values of variables along a nonlinear perfect foresight transition to a 20% higher debt-to-GDP ratio over 20 years. A period in the graphs is one year. The y-axis represent either percentage points differences or percentage changes compared to the baseline balanced growth path from which the transition starts.

Figure E.3 Nonlinear perfect foresight transition to the debt-to-GDP ratio of 1.2



(a) Labor income tax τ_t^L **(b)** Labor supply N_t **(c)** Capital stock $\tilde{K}_t$ **(d)** Growth rate g_t

Note: The figure illustrates the values of variables along a nonlinear perfect foresight transition to a 80% higher debt-to-GDP ratio over 20 years. A period in the graphs is one year. The y-axis represent either percentage points differences or percentage changes compared to the baseline balanced growth path from which the transition starts.